THE SIMPLE CALCULATION OF ELECTRICAL TRANSIENTS

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THE SIMPLE CALCULATION OF ELECTRICAL TRANSIENTS

An Elementary Treatment of Transient Problems in Linear Electrical Circuits, by Heaviside's Operational Method

by

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PREFACE

THIS BOOK is based upon a course of lectures given to engineers of the British Thomson-Houston Co. Ltd., and has been strongly influenced by their desires. Judging from them, it seems that most electrical engineers are aware that electrical transient problems can be solved by a mathematical method named after Oliver Heaviside; but the very fact that the method is mathematical puts it into a field in which many of them have practised little since their college days. The aim of this book is therefore to supply a link between the algebra and calculus of our youth and the circuit problems of our working life.

The book has been written throughout with a bias towards the physical or engineering side; the question discussed is not 'How do I solve this equation?' but 'How does this circuit work?' There must indeed be a sound mathematical foundation, if the reader is to avoid the pitfalls which beset a too-practical approach; but this necessary mathematics has been introduced gently, to reassure those readers whose practice in the subject is not of very recent date. As little previous knowledge as possible has been assumed, but some knowledge which must be taken for granted has been summarized in the last four Appendices.

The ground covered by the book is limited to the behaviour of linear circuits with lumped elements—a statement which is amplified and explained at the beginning of Chapter I. Within the field so limited the treatment is thorough, in the sense that I have not consciously omitted any important class of problem. Occasionally this policy has involved the discussion of somewhat abstruse ideas, such as the 'stability' considered in Chapter VI; especial care has been taken to present these parts in as simple a manner as possible. The Heaviside method has its own subtle difficulties, especially when it is applied to circuits which are not 'dead' to start with. I have not always found these difficulties dealt with very clearly in the literature of the subject, so I have tried to ensure that the exposition of them is as simple and methodical as I could make it.

No new step in the method has been introduced without being followed by one or more fully worked-out examples of it; and all these examples have been drawn from real live engineering or

laboratory problems. The threads of the argument have been gathered together in a final chapter consisting entirely of worked-out examples, which between them illustrate most of the principal themes of the book.

My thanks must first be accorded to my working colleagues, too numerous to name, who encouraged me to write the book and have helped me all through by their sustained interest and friendly criticism. The encouragement given by Mr F. H. Clough, C.B.E., Assistant Chief Engineer of the B.T.H. Co. Ltd., has been of especial value to me, and so has the help of Mr H. A. Price-Hughes, who has arranged for the proper drawing of the diagrams. Among previous writers on this subject, my former teacher, Dr Harold Jeffreys, F.R.S., has helped me most, as any reader will discover if he reads Chapter I of his *Operational Methods in Mathematical Physics* alongside Chapters II and III of the present work. Finally, I am very grateful to my friend Mr J. W. Boag, who has checked all the algebra and arithmetic; to my wife, who has typed the whole manuscript; and to the staff of the University Press, for their care in the printing.

G. W. C.

January 1944

CHAPTER I

INTRODUCTORY

1·1. The reader of this book is presumed to be already acquainted with electrical circuits; so he will realize that, in all their varied complexity, they are built up from a very small number of basic elements. Resistance, self-inductance, mutual inductance, and capacitance—these four exhaust the list. We shall be discussing circuits in which these four are variously combined; but the field which we shall cover will be narrowed in two ways.

(1) The circuit elements will be *linear*. This means that the voltage across a resistance bears a constant ratio to the current; the voltage across an inductance bears a constant ratio to the rate of change of current; the voltage across a capacitance bears a constant ratio to the charge. In practice, this signifies the exclusion of elements like these:

'Non-linear' resistance materials, such as are used in surge diverters; in these the current increases much faster than the voltage.

Electric arcs.

Inductances in which iron saturation is too great to be ignored.

(2) The circuit elements will be *lumped*. Thus, we shall have to do with condensers, which may be treated as blocks of capacitance, or with coils, which may be treated as blocks of self-inductance; but not with such things as cables, every millimetre of which contains a tiny bit of self-inductance in the conductor, and also a tiny bit of capacitance between the conductor and the screen or sheath. A cable is a typical example of a circuit whose elements are *distributed*; the behaviour of such circuits constitutes a more advanced branch of our subject.

It would be more exact to say 'The circuit elements can be represented as being lumped', for some circuits with distributed elements are equivalent for most of our purposes to circuits with lumped elements. Thus, each turn of a coil has self-inductance, mutual inductance to the other turns, and resistance; yet the whole coil is equivalent to a block of self-inductance in series with a block of resistance. It will be seen that the essential difference

between such a coil and a cable is that, in the coil, the current is the same (at a given moment) in every part; in a cable, it is not.

When non-linear circuits and distributed circuits have been excluded from our purview, the circuits which remain are neither uncommon nor contemptibly simple. In the field of power engineering alone, they comprise a great variety of power, testing and control circuits; exactly that class of circuit, in fact, whose behaviour under steady A.C. conditions is learned by every electrical engineering student. The reader will remember how, when each self-inductance L in a circuit has been treated as an impedance $Lj\omega$, and each capacitance C as an impedance $1/Cj\omega$, the two components of both voltage and current in each part of the circuit may be calculated by a method just like the method used for D.C. circuits. We owe this method to C. P. Steinmetz (1865–1923), and by introducing it he enabled engineers to solve A.C. circuit problems without continual recourse to the drawing-board.

This book treats of an analogous method for dealing with circuits under transient conditions, such as occur during the period of readjustment after a switch has been closed or opened. It is not much more difficult than the Steinmetz method, but it is by no means so widely known. It was devised by a man who used it on the problems of telegraphic transmission—Oliver Heaviside (1850–1925). He attained his results in a half-intuitive manner, which opened his methods to suspicion from the more orthodox mathematicians of his time. Heaviside's obscure and disorderly writings are enlivened by sarcasm directed at these misguided rigorists, such as 'Whether good mathematicians, when they die, go to Cambridge, I do not know'. However, Cambridge eventually returned good for evil by the hand of T. J. I'A. Bromwich (1875–1931), who showed that Heaviside's results really could be soundly based on mathematical reasoning. The same thing was done simultaneously by K. W. Wagner of Berlin; independently, for the date was 1916, and the native countries of the two workers were at war.

1·2. A pair of simple examples will show the parallelism between the Steinmetz and the Heaviside method.

(1) An alternating voltage V is applied to the circuit shown in Fig. 1. What will be the voltage v across the condenser?

Representing the condenser as an impedance $1/Cj\omega$, we see that

$$\frac{v}{V} = \frac{1/Cj\omega}{R+1/Cj\omega}$$

$$= \frac{1}{1+jRC\omega}$$

$$= \frac{1-jRC\omega}{1+(RC\omega)^2};$$

so v has the component $V/1+(RC\omega)^2$ in phase with V, and the component $-RC\omega V/1+(RC\omega)^2$ in quadrature with it.

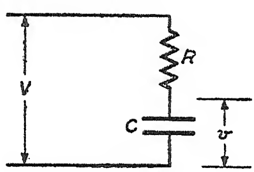

Fig. 1.

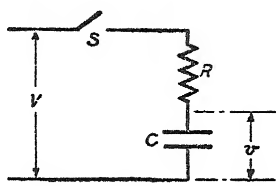

Fig. 2.

(2) In the circuit of Fig. 2, the switch S is closed, the condenser having initially no charge. A D.C. voltage V is thereby applied to the circuit. Find how the voltage v on the condenser increases.

In the Heaviside method of solving this problem, the condenser is regarded as an impedance $1/Cp$. Then

$$\frac{v}{V} = \frac{1/Cp}{R+1/Cp}$$

$$= \frac{1}{RCp+1}$$

$$= 1 - \frac{p}{p+1/RC}.$$

In other words, $$v = V - \frac{p}{p+1/RC}V.$$

Reference to a table of standard forms, such as is given in Appendix I, shows that

$$\frac{p}{p+1/RC}V$$

is to be interpreted as meaning

$$V \times e^{-t/RC},$$

where t denotes time from the instant of closing the switch. Therefore

$$v = V\{1-e^{-t/RC}\},$$

which gives the value of v at every instant after the time $t = 0$ when the switch was closed.

Simple though this process is, it is unwise to learn it by rote, as a child learns long division. The foundations of it need to be understood, if it is to be applied without fear of pitfalls. To explain these foundations as simply as possible is the task of the next few chapters; then we shall go on to some practical applications.

CHAPTER II

TRANSIENT CONDITIONS IN THE SIMPLEST CIRCUITS

2·1. The Differential Equation of a Circuit.

Consider further the circuit of Fig. 2. At every instant of time (denoted by t), after the switch has been closed, the charge q on the condenser is related to its voltage v by the equation

$$q = Cv.$$

Therefore the rate of increase of $q = C \times$ the rate of increase of v, since (as has been stated once for all in § 1·1) C does not vary: in other words

$$\frac{dq}{dt} = C\frac{dv}{dt}.$$

But the rate of growth of the charge is equal to the current which is flowing into one plate of the condenser and out of the other. Passing through the resistance R, this current gives rise to a voltage $R\,dq/dt$, or $RC\,dv/dt$. Now, at every instant,

Resistance voltage + Condenser voltage = Applied voltage V.

Therefore
$$RC\frac{dv}{dt}+v = V.$$

This equation, which expresses the behaviour of the circuit, is called the *differential equation* of the circuit; the *solution* of this equation is a relation between v and t, which enables the value of v at each instant t to be calculated. If the condenser started off already charged to a voltage v_0, we should expect to find a different solution from that corresponding to an initial state of complete discharge; and in fact no problem whose differential equation involves derivatives of the first order (dv/dt, but not d^2v/dt^2, d^3v/dt^3, ...) can be solved without a knowledge of *one* such starting-point (or its equivalent).

2·2. Solution of the Differential Equation.

Before proceeding to solve the equation

$$RC\frac{dv}{dt}+v = V,$$

let us rewrite it in a form which can embrace many other problems besides this one. To begin with, the applied voltage V was taken as a D.C. voltage for simplicity; but it might have been an alternating voltage, or even some more complicated shape. All these are covered if, instead of the constant V, we write a function of t, say $f(t)$. Again, we might have wanted to calculate charge, or current, instead of voltage; we cover all these possibilities if we write a non-committal u to stand for any of them. So the final form of the equation will be

$$a\frac{du}{dt}+bu=f(t),$$

where a and b are constants. We will seek that solution which makes $u=u_0$ when $t=0$.

We shall use the letter Q to denote the operation of integrating with respect to t, between the limits 0 and t; that is, $Q\phi(t)$ is a shorthand way of writing $\int_0^t \phi(t)\,dt$. Therefore, for instance,

$$Qt=t^2/2,$$

$$Qe^t=e^t-1,$$

and so forth. We perform the operation Q on every term of our equation, so as to get

$$Q\left(a\frac{du}{dt}\right)+Q(bu)=Qf(t),$$

and proceed to consider the meaning of each of these terms.

(1) $$Q\left(a\frac{du}{dt}\right)=\int_0^t a\frac{du}{dt}\,dt$$

$$=a\int_0^t \frac{du}{dt}\,dt$$

$$=a(u-u_0).$$

(2) $$Q(bu)=\int_0^t bu\,dt$$

$$=bQu.$$

We do not yet know u as a function of t, so we leave this term in the last-named form. Note that it is permissible to change the relative positions of Q and the constant b.

$$Qf(t) = \int_0^t f(t)\,dt, \tag{3}$$

which can be evaluated if necessary, but will be left for the moment in its original form.

Rewriting the equation in the light of these statements, we obtain

$$a(u-u_0)+bQu = Qf(t),$$

or

$$u = u_0+\frac{1}{a}Qf(t)-\frac{b}{a}Qu.$$

We shall now start to write α instead of b/a.

The first two terms of the right-hand side are known, but not the last term. However, for u in the last term we can substitute the expression

$$u_0+\frac{1}{a}Qf(t)-\alpha Qu,$$

so as to get

$$u = u_0+\frac{1}{a}Qf(t)-\alpha Q\{u_0+\frac{1}{a}Qf(t)-\alpha Qu\}.$$

If the same substitution is performed repeatedly, we get

$$u = u_0+\frac{1}{a}Qf(t)-\alpha Q\left\{u_0+\frac{1}{a}Qf(t)-\alpha Q\left[u_0+\frac{1}{a}Qf(t)-\dots\right]\right\},$$

so that the unknown term is, as it were, pushed farther and farther away.

Now $\alpha Q\alpha Qu_0$ is the same thing as $\alpha^2Q^2u_0$, where Q^2 means that the operation Q must be performed twice. Therefore, collecting together all the u_0 terms and all the $f(t)$ terms, we get

$$u = (1-\alpha Q+\alpha^2Q^2-\alpha^3Q^3+\dots)u_0+(Q-\alpha Q^2+\alpha^2Q^3-\dots)\left[\frac{1}{a}f(t)\right].$$

Every term of these expressions can be evaluated by integrating once, twice, thrice, four times, ...; so here is an equation which makes it possible to work out u in terms of t.

The u_0 bracket leads readily to a simple result. For

$$\alpha Q u_0 = \alpha \int_0^t u_0\,dt \quad = u_0 . \alpha t,$$

$$\alpha^2 Q^2 u_0 = \alpha \int_0^t u_0 \alpha t\,dt = u_0 . \alpha^2 t^2/2\,!,$$

$$\alpha^3 Q^3 u_0 \qquad\qquad = u_0 . \alpha^3 t^3/3\,!,$$

and so on. Therefore

$$(1-\alpha Q+\alpha^2 Q^2-\alpha^3 Q^3+\ldots)\,u_0 = u_0\left(1-\alpha t+\frac{\alpha^2 t^2}{2\,!}-\frac{\alpha^3 t^3}{3\,!}+\ldots\right).$$

We recognize this series as the exponential one, whose sum is $e^{-\alpha t}$. Therefore

$$u = u_0 e^{-\alpha t}+(Q-\alpha Q^2+\alpha^2 Q^3-\ldots)\left[\frac{1}{a}f(t)\right].$$

There is no obvious way of finding the sum of the second series without knowing the exact form of $f(t)$; so at the present stage we can only hope that it will usually prove as recognizable as the first one. Mathematicians might be seized at this point with a horrible misgiving that the series may be 'divergent' (or meaningless); but we can be fortified by the reflection that our problem, being a physical problem, *must* have a solution which means something.

2·3. Example. Solution of a Circuit by the Q method.

Let us apply this method to the circuit of Fig. 2, whose differential equation was shown to be

$$RC\frac{dv}{dt}+v = V;$$

but let us suppose that, prior to the closing of the switch, the condenser was charged to a voltage v_0.

The equation may be written

$$\frac{dv}{dt}+\alpha v = \alpha V,$$

where $\alpha = 1/RC$. The operation Q, performed on every term, gives

$$v-v_0+\alpha Q v = \alpha Q V,$$

or

$$v = v_0+\alpha Q V-\alpha Q v;$$

and the method already explained leads on to

$$\begin{aligned} v &= v_0 + \alpha Q V - \alpha Q \{v_0 + \alpha Q V - \alpha Q [v_0 + \alpha Q V - \ldots]\} \\ &= (1 - \alpha Q + \alpha^2 Q^2 - \ldots) v_0 + (\alpha Q - \alpha^2 Q^2 + \alpha^3 Q^3 - \ldots) V \\ &= v_0 e^{-\alpha t} + (\alpha Q - \alpha^2 Q^2 + \alpha^3 Q^3 - \ldots) V. \end{aligned}$$

We observe with joy that the second term also leads to a recognizable series, since V is constant; namely

$$V\left(\alpha t - \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} - \ldots\right),$$

which is an exponential series shorn of its first term, and equals

$$V(1 - e^{-\alpha t}).$$

So the solution comes to be

$$v = v_0 e^{-\alpha t} + V(1 - e^{-\alpha t}),$$

where $\alpha = 1/RC$. This shows that the relation between v and t is of the form shown in Fig. 3; starting at the initial value v_0, the voltage climbs up to the full value V.

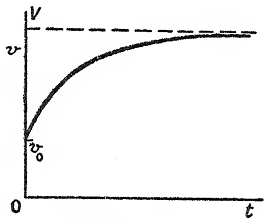

Fig. 3.

I hasten to assure the reader that the Heaviside method does *not* involve this ponderous process, except in the earliest stage of learning it.

2·4. Behaviour of the Operator Q.

The method just described is called an *operational* method, because it is carried out by means of the 'operator' Q. (There is nothing more abstruse about the use of the symbol Q to denote the operation of integrating, than about the use of the symbol $\times$ to denote the operation of multiplying.) Hitherto we have treated Q with respect, remembering always that it is not a number but an operator; but now we shall discover that great abbreviation is possible if we appreciate that it can in fact be treated very nearly as a number.

In solving the equation

$$a\frac{du}{dt} + bu = f(t),$$

we reached almost immediately the stage

$$u = u_0 + \frac{1}{a} Q f(t) - \alpha Q u.$$

Suppose we had solved this equation for u, treating Q as a number. We should have obtained

$$u = \frac{1}{1+\alpha Q} u_0 + \frac{Q}{1+\alpha Q}\left[\frac{1}{a} f(t)\right].$$

Now if Q were a number, $1/1+\alpha Q$ could be expanded by the Binomial Theorem (see Appendix II), with the result

$$\frac{1}{1+\alpha Q} = 1-\alpha Q+\alpha^2 Q^2-\alpha^3 Q^3+\ldots.$$

This would have led us immediately to the equation

$$u = (1-\alpha Q+\alpha^2 Q^2-\ldots)\, u_0+(Q-\alpha Q^2+\alpha^2 Q^3-\ldots)\left[\frac{1}{a} f(t)\right],$$

which we got before, with much more labour. The method leads to the right answer, so there must be something in it. Why are we justified in treating Q in this way? and do any pitfalls lurk to entrap us when we do it?

If u and v are two functions of t,

$$\begin{aligned} Q(u+v) &= \int_0^t (u+v)\,dt \\ &= \int_0^t u\,dt + \int_0^t v\,dt \\ &= Qu+Qv, \end{aligned}$$

just as if Q were a number. Similarly

$$Q^m Q^n u = Q^{m+n} u,$$

and, if α is a constant,

$$Q(\alpha u) = \alpha Q u.$$

But $Q(uv)$ is *not* equal to uQv: for instance, $Q(t.t) = Qt^2 = t^3/3$, whereas $t.Qt = t^3/2$.

Thus we may say that Q obeys the ordinary laws of algebra, *except that it must not change places with a variable*. We may treat Q as a number, provided that we take care that its position relatively to variable terms is not altered. For example, we must not change $Q(t \sin t)$ into $t.Q \sin t$, for they do not mean the same thing.

There is therefore a justification for the rapid transition from

$$u = u_0 + \frac{1}{a} Qf(t) - \alpha Qu,$$

or
$$(1+\alpha Q)\,u = u_0 + \frac{1}{a} Qf(t),$$

to $\quad u = (1-\alpha Q+\alpha^2Q^2-\ldots)\,u_0+(Q-\alpha Q^2+\alpha^2Q^3-\ldots)\left[\frac{1}{a}f(t)\right].$

Neither in the division of both sides by $(1+\alpha Q)$, nor in the replacement of $1/1+\alpha Q$ by its binomial expansion, is there any violation of the rule that Q must not change places with a variable.

2·5. Heaviside's Operator p.

We are on the verge of being able to write down a simple rule for solving any circuit of this type; but before doing so we have to bid farewell to our operator Q, since it is not in general use. The operator used by Heaviside and his successors is denoted by the letter p, and we shall define it by the equation

$$p = 1/Q.$$

For the moment, the reader may think of this change as being simply a change of notation; for reasons not yet revealed, we prefer $1/p$ to Q, as a notation for the operation $\int_0^t \ldots\, dt$. So far, we have attached a meaning to $Q, Q^2, \ldots$, but not to $1/Q, 1/Q^2, \ldots$; therefore we can attach a meaning to $1/p, 1/p^2, \ldots$, but not (as yet) to $p, p^2, \ldots$. Q is more convenient in proving the method, but p is more convenient in applying it.

All that has been said about the legitimacy of treating Q as a number applies equally to p; it obeys the ordinary laws of algebra, except that it must not change places with a variable.

2·6. Rules for Solution of a Simple Electrical Circuit.

The formal solution

$$u = \frac{1}{1+\alpha Q} u_0 + \frac{Q}{1+\alpha Q}\left[\frac{1}{a}f(t)\right]$$

now takes the form

$$u = \frac{p}{p+\alpha} u_0 + \frac{1}{p+\alpha}\left[\frac{1}{a}f(t)\right]$$

$$= \frac{ap}{ap+b} u_0 + \frac{1}{ap+b} f(t).$$

We may therefore write down the following rules for the solution of any circuit which is simple enough to lead to a differential equation of the form

$$a\frac{du}{dt}+bu=f(t).$$

(It is not necessary to define what circuits lead to equations of this type, since we shall almost immediately be proceeding to more complicated circuits.)

(1) Write down the differential equation of the circuit for the quantity desired (e.g. voltage), in the form

$$a\frac{du}{dt}+bu=f(t).$$

(2) On the left-hand side, replace d/dt by p.

(3) To the right-hand side, add the result of dropping the u-term from the left-hand side and changing (pu) into (pu_0).

(4) Solve the resulting equation algebraically for u (treating p as a number), and evaluate the result by expanding in powers of $1/p$ and interpreting $1/p$ as $\int_0^t \ldots dt$.

The first stage may afford difficulty to some. Later on we shall show that stages (1) and (2) can both be replaced by a process closely resembling the Steinmetz method for solving A.C. circuits under steady conditions. For the present, we merely recall that

Voltage and current in a resistance R are related by

$$v = Ri;$$

Voltage and current in an inductance L are related by

$$v = L\,di/dt;$$

Voltage and current in a capacitance C are related by

$$i = C\,dv/dt;$$

while in a mutual inductance M, whose primary and secondary coils have self-inductances L_1, L_2,

Primary voltage $$v_1 = L_1\frac{di_1}{dt}+M\frac{di_2}{dt},$$

and Secondary voltage $$v_2 = M\frac{di_1}{dt}+L_2\frac{di_2}{dt}.$$

These equations of course involve certain assumptions about the relative directions of current and voltage—if one is reversed, a negative sign will creep in.

2·7. Standard Operational Forms.

We do not in practice carry out every time the expansion in powers of $1/p$. We do it once for all, and record the result for future reference as a 'Standard Form'. (See Appendix I.) It will be seen that the solution of this type of circuit involves two such forms; namely

$$\frac{p}{p+\alpha} \text{ operating on a constant, } u_0,$$

and
$$\frac{1}{p+\alpha} \text{ operating on a function of } t.$$

We proceed to find out the general meaning of these. Now

$$\begin{aligned}\frac{p}{p+\alpha}.1 &= \frac{1}{1+\alpha Q}.1\\ &= (1-\alpha Q+\alpha^2Q^2-\ldots).1\\ &= e^{-\alpha t},\end{aligned}$$

as has been seen already. Also $\dfrac{1}{p+\alpha}f(t)$ is that solution of the equation

$$\frac{du}{dt}+\alpha u = f(t)$$

which makes $u = 0$ when $t = 0$. To find this, multiply both sides of the equation by $e^{\alpha t}$; we get

$$e^{\alpha t}\frac{du}{dt}+\alpha e^{\alpha t}u = e^{\alpha t}f(t),$$

or
$$\frac{d}{dt}(ue^{\alpha t}) = e^{\alpha t}f(t),$$

or
$$ue^{\alpha t} = \int_0^t e^{\alpha t}f(t)\,dt,$$

since $u = 0$ when $t = 0$;

or
$$u = e^{-\alpha t}\int_0^t e^{\alpha t}f(t)\,dt.$$

Thus $$\frac{1}{p+\alpha}f(t) = e^{-\alpha t}\int_0^t e^{\alpha t}f(t)\,dt;$$

or, in words,

'The operation $1/p+\alpha$ means: Multiply by $e^{\alpha t}$, integrate the product from 0 to t, and multiply the result by $e^{-\alpha t}$.'

This last result gives a simple expression for something which hitherto we have known only as an infinite series, namely

$$(Q-\alpha Q^2+\alpha^2Q^3-\ldots)f(t). \qquad \text{(See § 2·2.)}$$

Instead of having to integrate $f(t)$ once, twice, thrice, ..., hoping to discover some rhyme or reason about the process, we now have to perform a single integration only. In Appendix I will be found the result of performing the operation $1/p+\alpha$ on certain commonly encountered functions of t.

This chapter will be concluded with some practical examples.

2·8. Examples. Solution of the Simplest Type of Circuit.

Example 1. The field winding of an alternator has a self-inductance L henries, and a resistance R ohms. It is passing a current of i_0 amps. from a D.C. supply (Fig. 4). By the operation of a field suppression circuit, the winding is suddenly disconnected from the supply and short-circuited on itself. Find how the current now varies.

The operation is carried out by moving the switch shown in Fig. 4 from position A to position B; the circuit must be closed at B before it is opened at A. There will also be a parallel circuit (not shown) for receiving the current which is still wanting to flow from the D.C. supply.

At any instant t seconds after the switching, suppose the current to be i amps. The inductive voltage in the field winding is $L\,di/dt$; the resistance voltage, Ri. Since the winding is short-circuited,

$$\text{Voltage on } L + \text{voltage on } R = 0,$$

that is, $$L\frac{di}{dt}+Ri = 0;$$

and, when $t=0$, $$i = i_0.$$

This is the differential equation of the circuit. The p-equation, by the rules given in § 2·6, is

$$(Lp+R)\,i = Lpi_0.$$

Therefore
$$i = \frac{Lp}{Lp+R} i_0$$
$$= \frac{p}{p+(R/L)} i_0$$
$$= i_0 e^{-Rt/L}.$$

Thus the current collapses according to an exponential law. It has fallen to half its original value when t has the value given by the equation
$$e^{-Rt/L} = \tfrac{1}{2},$$
or
$$e^{Rt/L} = 2,$$
or
$$Rt/L = \log_e 2 = 0{\cdot}693,$$
or
$$t = 0{\cdot}693 L/R.$$

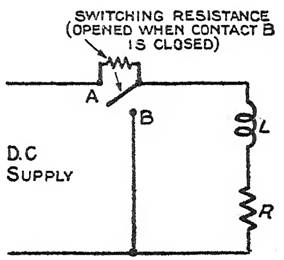

Fig. 4.

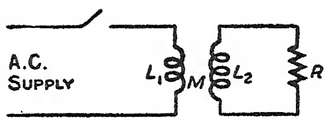

Fig. 5.

Example 2. A transformer, whose windings have self-inductances L_1, L_2 henries and a mutual inductance M henries, is connected on its secondary side to a resistance load of R ohms; the resistances of the windings are neglected (Fig. 5). A switch is closed on the primary side, connecting the transformer to an A.C. voltage supply of peak value V and frequency $\Omega/2\pi$, at the instant of voltage maximum. Before the switch was closed, the transformer was dead. Find the voltage across the load resistance R.

If the primary and secondary currents are i_1 and i_2 amps., we have
$$L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = \text{Voltage applied to primary terminals}$$
$$= V \cos \Omega t,$$

since the voltage has to be a maximum when $t = 0$. Also, considering the secondary circuit,

$$M\frac{di_1}{dt}+L_2\frac{di_2}{dt}+Ri_2 = 0,$$

since there is no source of voltage in this circuit (apart from the transformer voltage, which is represented by the $M\,di_1/dt$ term).

We are only interested in the secondary circuit; so we eliminate i_1, by multiplying the former equation by M/L_1 and subtracting it from the latter. This gives

$$\left(L_2-\frac{M^2}{L_1}\right)\frac{di_2}{dt}+Ri_2 = -\frac{M}{L_1}V\cos\Omega t.$$

Replacing i_2 by v/R, where v is the desired secondary voltage, we obtain

$$\left(\frac{L_1L_2-M^2}{L_1R}\right)\frac{dv}{dt}+v = -\frac{M}{L_1}V\cos\Omega t.$$

This is the differential equation.

The p-equation is therefore

$$\left\{\left(\frac{L_1L_2-M^2}{L_1R}\right)p+1\right\}v = -\frac{M}{L_1}V\cos\Omega t,$$

for, with $v = 0$ initially, there is no v_0 term to add. Therefore

$$v = \frac{1}{\left(\dfrac{L_1L_2-M^2}{L_1R}\right)p+1}\left(-\frac{M}{L_1}V\cos\Omega t\right)$$

$$= \frac{1}{p+\alpha}\left(-\frac{\alpha MV}{L_1}\right)\cos\Omega t,$$

where $$\alpha = \frac{L_1R}{L_1L_2-M^2}.$$

Reference to Appendix I (form 23) shows that this is to be interpreted as

$$v = -\frac{MV}{L_1}\frac{\alpha}{\alpha^2+\Omega^2}\{\alpha\cos\Omega t+\Omega\sin\Omega t-\alpha e^{-\alpha t}\}.$$

The behaviour of v in a typical case is shown in Fig. 6. The last term in the expression is a transient one, and when it has died out we are left with an alternating voltage only.

The constant α may be written R/L, where

$$L = L_2 - \frac{M^2}{L_1}.$$

L is the short-circuit reactance of the transformer, viewed from the secondary side; and $(-MV/L_1)$ is the peak value of the e.m.f. referred to the secondary side.

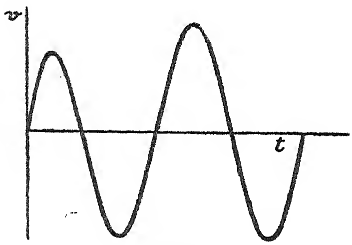

Fig. 6.

If the steady A.C. voltage across R is worked out by the Steinmetz method, assuming series inductance L and e.m.f. $(-MV/L_1)$, the result will be found to be

$$-\frac{MV}{L_1}\frac{\alpha}{\alpha^2+\Omega^2}(\alpha - j\Omega),$$

which is of course equivalent to the first two terms of the expression which we have worked out for v.

CHAPTER III

TRANSIENT CONDITIONS IN MORE COMPLICATED CIRCUITS

3·1. Significance of the Operator p.

Before going on to more complicated circuits, we pause to ask a question: 'If $Q = \int_0^t dt$, and $p = 1/Q$, what is p? What physical meaning, if any, does it possess?'

Since p replaces d/dt in the equations, it is tempting to suppose that p means d/dt; and indeed that meaning is often attributed to it. But this interpretation may easily lead us into trouble. For instance, in Example 1 at the end of the last chapter (§ 2·8), the auxiliary p-equation was

$$(Lp+R)\,i = Lpi_0;$$

and if p means d/dt, the right-hand side is zero, so that we are led to the odd conclusion that the decay of current in a field suppression circuit is not affected by the current you start with!

A truer insight may be obtained, if we observe that the equation

$$p = 1/Q$$

means that p is the operator which undoes the operation Q. Does d/dt do this? Assuredly

$$\frac{d}{dt}Qf(t) = f(t);$$

but

$$Q\frac{d}{dt}f(t) = \int_0^t f'(t)\,dt$$

$$= f(t) - f(0).$$

Thus d/dt undoes Q, if, and only if, it acts after Q (apart from the special case $f(0) = 0$). In other words, p can only be identified with d/dt if the differentiations are performed *after* the integrations. The wisest course, however, is to remember that p means $1/Q$, and forget its identification with d/dt. Only seldom is the latter necessary.

3·2. Circuits of the Next Stage of Complexity.

In Chapter II we considered exclusively such circuits as would lead to differential equations of form

$$a\frac{du}{dt}+bu=f(t).$$

In practice, this meant simple combinations of resistance with either inductance or capacitance. If we consider two simple resistance-inductance circuits with mutual inductance between them, we have a form which makes a convenient stepping-stone to a general theory.

In the circuit of Fig. 7 the switch is closed at time $t=0$ so that the circuit (hitherto dead) is connected to a 1-volt supply. How do the primary and secondary currents behave?

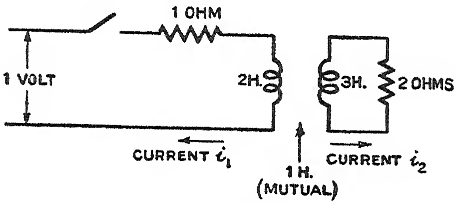

Fig. 7.

Let the currents be i_1 and i_2; then, just as in Example 2 at the end of the last chapter (§ 2·8), the differential equations of the circuit are

$$2\frac{di_1}{dt}+\frac{di_2}{dt}+i_1=1,$$

$$\frac{di_1}{dt}+3\frac{di_2}{dt}+2i_2=0.$$

Now if the initial values of i_1 and i_2 were i_{10} and i_{20}, the auxiliary p-equations (written down by rules exactly analogous to those laid down for a single variable) would be

$$(2p+1)\,i_1+pi_2=1+2pi_{10}+pi_{20},$$

and
$$pi_1+(3p+2)\,i_2=\qquad pi_{10}+3pi_{20}.$$

In this particular case, i_{10} and i_{20} are zero; so the p-equations are simply

$$(2p+1)\,i_1+pi_2=1,$$

$$pi_1+(3p+2)\,i_2=0.$$

Solving for i_1, treating p as a number, we get

$$i_1 = \frac{3p+2}{(2p+1)(3p+2)-p^2} \cdot 1$$

$$= \frac{3p+2}{5p^2+7p+2} \cdot 1.$$

As it stands, this expression is not a standard form. The way to interpret it is to split it into 'partial fractions' which are standard

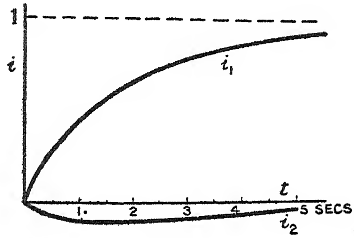

Fig. 8.

forms. Rules for this will be given presently, so we will simply write down the result here. The above expression for i_1 is the same as

$$i_1 = \frac{3p+2}{(5p+2)(p+1)} \cdot 1,$$

and the reader may verify that this is algebraically equivalent to

$$i_1 = \left(1 - \frac{2}{3}\frac{p}{p+(\frac{2}{5})} - \frac{1}{3}\frac{p}{p+1}\right) \cdot 1$$

$$= 1 - \tfrac{2}{3}e^{-2t/5} - \tfrac{1}{3}e^{-t}.$$

Similarly $\qquad i_2 = \tfrac{1}{3}(e^{-t} - e^{-2t/5}).$

The currents therefore vary in the manner shown in Fig. 8. As is to be expected, i_1 tends towards the D.C. value of 1 amp., whereas i_2 settles down to zero when the effects of the switching operation have died away.

3·3. The Differential Equation of Any Linear Circuit.

In solving this last example, we wrote down a pair of differential equations which contained both i_1 and i_2, and proceeded immedi-

ately to solve them by the p method. But we might instead have got rid (for instance) of i_2, and so obtained a differential equation for i_1 only; this would, in fact, have been

$$5\frac{d^2i_1}{dt^2}+7\frac{di_1}{dt}+2i_1=2.$$

We observe that this is an equation of the general form

$$a\frac{d^2u}{dt^2}+b\frac{du}{dt}+cu=f(t);$$

and we recall that the circuits of the last chapter led to the form

$$a\frac{du}{dt}+bu=f(t).$$

It begins to dawn upon us that the differential equations associated with *any* circuit with linear lumped elements are likely to be of the form

$$a\frac{d^nu}{dt^n}+b\frac{d^{n-1}u}{dt^{n-1}}+\ldots+l\frac{du}{dt}+mu=f(t),$$

and indeed this does prove to be the case.

These equations are all of the form known as *linear differential equations with constant coefficients*; and if the highest-order derivative occurring in the equation is d^nu/dt^n, the equation is said to be of the nth order. Our chief interest is in the circuit, not in the differential equation; but it saves a lot of circumlocution if we write 'a circuit of the nth order', instead of 'a circuit whose associated differential equations are of the form

$$a\frac{d^nu}{dt^n}+\ldots+l\frac{du}{dt}+mu=f(t)\text{'}.$$

It now becomes plain that the circuits solved in Chapter II were circuits of the first order; and that the circuit of Fig. 7 is a second-order circuit.

3·4. Further Standard Forms.

These higher-order circuits are capable of leading to p-expressions of other forms than the two already discussed: as, for

instance, $\frac{p^2}{p^2+\alpha^2}.1$. These can all be interpreted by expansion in powers of $1/p$; for instance,

$$\frac{p^2}{p^2+\alpha^2}.1 = \frac{1}{1+(\alpha^2/p^2)}.1$$
$$= \left(1-\frac{\alpha^2}{p^2}+\frac{\alpha^4}{p^4}-\ldots\right).1$$
$$= 1-\frac{\alpha^2t^2}{2!}+\frac{\alpha^4t^4}{4!}-\ldots$$
$$= \cos\alpha t.$$

A list of such standard forms will be found in Appendix I. Some of the more complex ones may be neatly derived from simpler ones like this:

We know, for instance, that $\frac{p}{p+\alpha}.1 = e^{-\alpha t}$.

Differentiate both sides of this equation with respect to α, treating p and t as constant. We get

$$\frac{-p}{(p+\alpha)^2}.1 = -te^{-\alpha t},$$

or $$\frac{p}{(p+\alpha)^2}.1 = te^{\alpha t}, \text{ a new standard form.}$$

It may seem strange to treat the one constant in the equation as a variable; however, the method leads to the right answer, so it is not so unsound as it looks.

3·5. Method of dealing with Operations upon $f(t)$.

It will be seen that most of the standard forms in the list are concerned with interpreting an expression of the form $F(p).1$; but we know from experience with first-order circuits that we may also have to interpret expressions like $F(p).f(t)$. In very many circuit problems, however, $f(t)$ is an elementary function such as a sine wave. For such functions, an operational equivalent may be immediately picked out from Appendix I; so that, if the equivalent of

$$f(t) \text{ is } \phi(p).1,$$

we may immediately write

$$F(p).f(t) = F(p).\phi(p).1;$$

and the combined function $F(p)\phi(p)$ can be split up into a number of standard forms of the kind in which a function of p operates upon 1.

For instance, if it were necessary to interpret

$$\frac{3}{2}\frac{1}{p^2+1}\sin 2t,$$

we should use Appendix I to write $\sin 2t$ in the form

$$\frac{2p}{p^2+4}.1.$$

Thus

$$\begin{aligned}\frac{3}{2}\frac{1}{p^2+1}\sin 2t &= \frac{3p}{(p^2+1)(p^2+4)}.1\\ &= \left(\frac{p}{p^2+1}-\frac{p}{p^2+4}\right).1\\ &= \sin t - \tfrac{1}{2}\sin 2t.\end{aligned}$$

3·6. Heaviside's Shifting Theorem.

A theorem which is sometimes used is here mentioned for reference only, in case the reader should find it named in the literature. If a function of p operates upon a function of t having an exponential factor, it is possible to remove the exponential by the use of the identity

$$F(p).\{e^{\alpha t}f(t)\} \equiv e^{\alpha t}F(p+\alpha).\{f(t)\}.$$

This can sometimes lead to a short cut, or to a method of deriving a fresh 'standard form'. But if the reader is prepared to take 'standard forms' for granted, he can safely leave it unlearnt.

3·7. Formation of Partial Fractions.

The reader will have gathered that it is necessary to split up complex algebraic functions of p into a number of expressions each simple enough to be a standard form. The following rules will enable this to be done.

We will suppose that we have to interpret such an expression as

$$\frac{b_m p^m + b_{m-1}p^{m-1} + \dots + b_1 p + b_0}{a_n p^n + a_{n-1}p^{n-1} + \dots + a_1 p + a_0} \cdot 1.$$

We always find in circuit problems that m is not greater than n.

It is shown in Appendix V that algebraic expressions, such as the top or bottom lines of this fraction, can be split into real factors:

either single linear factors $(p+\alpha)$,

or repeated linear factors $(p+\beta)^r$,

or single quadratic factors (p^2+ap+b),

or repeated quadratic factors $(p^2+cp+d)^s$.

In the last two cases $a^2 < 4b$, $c^2 < 4d$: for otherwise the expressions could be further split up into real factors $(p+\gamma)(p+\delta)$. Quadratic factors are only introduced when the linear ones are not real.

The partial fractions are found as follows.

(1) Rewrite the bottom line of the operational fraction in its real factors, thus:

$$\frac{b_m p^m + b_{m-1}p^{m-1} + \dots + b_1 p + b_0}{a_n(p+\alpha)\dots(p+\beta)^r\dots(p^2+ap+b)\dots(p^2+cp+d)^s\dots}.$$

(2) Equate this expression to the sum of a string of fractions:

for each single linear factor $\dfrac{\dots\dots}{(p+\alpha)\dots}$, assume a fraction of form $\dfrac{Ap}{p+\alpha}$;

for each repeated linear factor $\dfrac{\dots\dots}{(p+\beta)^r\dots}$, assume fractions of form $\dfrac{Bp}{p+\beta}+\dfrac{Cp}{(p+\beta)^2}+\dots+\dfrac{Gp}{(p+\beta)^r}$;

for each single quadratic factor $\dfrac{\dots\dots}{(p^2+ap+b)\dots}$, assume a fraction of form $\dfrac{Hp^2+Kp}{p^2+ap+b}$;

for each repeated quadratic factor $\dfrac{\dots\dots}{(p^2+cp+d)^s\dots}$, assume fractions of form

$$\frac{Lp^2+Mp}{p^2+cp+d}+\frac{Np^2+Pp}{(p^2+cp+d)^2}+\dots+\frac{Sp^2+Tp}{(p^2+cp+d)^s};$$

and in every case assume also a constant term W. Then it will be found that it is possible to choose numbers for $A, B, C, \ldots, W$, so that the sum of the string of fractions is equal to the original expression

$$\frac{b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0}{a_n(p+\alpha)\ldots(p+\beta)^r \ldots (p^2+ap+b)\ldots(p^2+cp+d)^s\ldots}.$$

There will be just enough disposable numbers $A, B, \ldots$ for this; neither too many nor too few.

There exists a general formula, known as Heaviside's Expansion Theorem, for writing down the function of t which emerges when any algebraic function of p operates upon 1. In practice it is less convenient than the method just given, and is only mentioned here in case the reader should find it named in the literature and desire to know what it is.

3·8. Examples of the Formation of Partial Fractions.

Example 1. The circuit of Fig. 7 led us to the expression

$$i_1 = \frac{3p+2}{5p^2+7p+2}.1$$

$$= \frac{3p+2}{5(p+\frac{2}{5})(p+1)}.1.$$

Let
$$\frac{3p+2}{5(p+\frac{2}{5})(p+1)} \equiv \frac{Ap}{p+\frac{2}{5}} + \frac{Bp}{p+1} + C.$$

Then (multiplying both sides by $5(p+\frac{2}{5})(p+1)$),

$$3p+2 \equiv 5Ap(p+1) + 5Bp(p+\tfrac{2}{5}) + 5C(p+\tfrac{2}{5})(p+1).$$

The simplest way to get A and B, in such a case as this where the factors are linear, is to give p particular values, which cause the disappearance of every group except one. Thus

$$p = -\tfrac{2}{5} \text{ gives } -\tfrac{6}{5}+2 = 5A(-\tfrac{2}{5})(\tfrac{3}{5}), \text{ whence } A = -\tfrac{2}{3};$$

$$p = -1 \text{ gives } -3+2 = 5B(-1)(-\tfrac{3}{5}), \text{ whence } B = -\tfrac{1}{3}.$$

The constant term C can always be obtained by writing $p = 0$; here, this gives $C = 1$. Therefore

$$i_1 = \left(-\frac{2}{3}\frac{p}{p+\frac{2}{5}} - \frac{1}{3}\frac{p}{p+1} + 1\right).1,$$

which is interpreted in the way already indicated.

Example 2. In § 3·5, we had to interpret the expression

$$\frac{3p}{(p^2+1)(p^2+4)}.1.$$

Following the rules, we assume

$$\frac{3p}{(p^2+1)(p^2+4)} \equiv \frac{Ap^2+Bp}{p^2+1}+\frac{Cp^2+Dp}{p^2+4}+E.$$

Writing $p = 0$ shows that $E = 0$. For the rest, multiply up, so as to obtain

$$3p \equiv (Ap^2+Bp)(p^2+4)+(Cp^2+Dp)(p^2+1).$$

The right-hand side contains terms in p^4, p^3, p^2, and p. Equating the coefficients of these with the corresponding coefficients on the left, we obtain

$$\begin{aligned} A+C &= 0, \\ B+D &= 0, \\ 4A+C &= 0, \\ 4B+D &= 3. \end{aligned}$$

The first and third give $A = C = 0$; the second and fourth give $B = 1$, $D = -1$. Therefore

$$\frac{3p}{(p^2+1)(p^2+4)}.1 = \left(\frac{p}{p^2+1}-\frac{p}{p^2+4}\right).1, \text{ as before.}$$

3·9. Circuits of the Second Order.

We have seen that there is a class of circuit whose differential equations have the form

$$a\frac{d^2u}{dt^2}+b\frac{du}{dt}+cu = f(t).$$

Just as, in the first-order circuit, we needed to know the starting value of *one* quantity, so in the second-order equation we must know *two* starting values (or their equivalent). For instance, it is sufficient if we know that $u = u_0$, $du/dt = u_1$, when $t = 0$.

The equation is easily split into two of the first order, by taking a new variable $v \equiv du/dt$. Then

$$\begin{aligned} a\frac{dv}{dt}+bv+cu &= f(t), \\ \frac{du}{dt}-v &= 0. \end{aligned}$$

The auxiliary p-equations, written down in the usual way, are

$$(ap+b)\,v+cu = f(t)+apu_1,$$

$$pu-v = pu_0.$$

Eliminating v algebraically, we find

$$(ap^2+bp+c)\,u = f(t)+(ap^2+bp)\,u_0+apu_1.$$

This can be interpreted by splitting each term into suitable partial fractions. We shall not go through the process here, for it will be performed in several examples at the end of this chapter.

3·10. Circuits of Any Order.

In exactly the same way we can solve a circuit of order n, given that, when $t = 0$, $u = u_0$, $du/dt = u_1$, $d^2u/dt^2 = u_2$, etc. It soon appears that the p-equation may be written down from the differential equation by the following rules.

(1) On the left-hand side, replace d/dt by p.

(2) On the right-hand side, leave $f(t)$ unchanged, but add terms in $u_0, u_1, u_2, \ldots$ obtained in the following manner:

- u_0 term. Drop the term which does not contain p from the left-hand side, and write u_0 for u;
- u_1 term. Divide the last named by p, drop the term which now does not contain p, and write u_1 for u_0;
- u_2 term. Divide the last named by p, drop the term which now does not contain p, and write u_2 for u_1;

and so on.

Thus, the third-order differential equation

$$a\frac{d^3u}{dt^3}+b\frac{d^2u}{dt^2}+c\frac{du}{dt}+du = f(t)$$

leads to the p-equation

$$(ap^3+bp^2+cp+d)\,u = f(t)+(ap^3+bp^2+cp)\,u_0+(ap^2+bp)\,u_1+apu_2.$$

The extension to simultaneous equations (involving two or more dependent variables $u, v, w, \ldots$) involves no new principle but is rarely required.

3·11. Examples. Circuits of Order Higher than the First.

Example 1. This circuit resembles that shown in Fig. 2, except that inductance L has been added (Fig. 9). The closing of the switch S applies to the circuit a D.C. voltage V; how does the condenser voltage v behave, assuming that it was zero to start with?

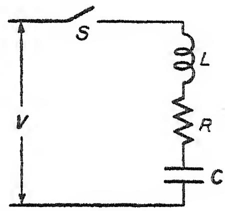

Fig. 9.

We have seen (§ 2·6) that the current and voltage in a condenser are related by the equation

$$i = C\frac{dv}{dt}.$$

Therefore the inductance voltage $= L\dfrac{di}{dt} = LC\dfrac{d^2v}{dt^2}$;

the resistance voltage $= Ri = RC\dfrac{dv}{dt}$;

the condenser voltage $= v$;

and the differential equation is

$$LC\frac{d^2v}{dt^2}+RC\frac{dv}{dt}+v = V.$$

We must now consider what are the initial values of v and dv/dt. The condenser is initially discharged; its voltage cannot give a leap when the switch is closed, for that would involve passing an infinite current through L and R; so $v_0 = 0$. Again, dv/dt is proportional to the current, and this cannot change suddenly because the voltage across L would then be infinite; so v_1, the initial value of dv/dt, is zero also. Hence the p-equation is

$$(LCp^2+RCp+1)\,v = V,$$

and

$$v = \frac{1}{p^2+\dfrac{R}{L}p+\dfrac{1}{LC}}\cdot\frac{V}{LC}.$$

The solution of this may take one of two forms, according as R^2 is greater or less than $4L/C$ (see Appendix V). If $R^2 > 4L/C$, the denominator has real factors $(p+\alpha)(p+\beta)$. If $R^2 < 4L/C$, the factors are complex, and the denominator may be written in the form $(p+\gamma)^2+\omega^2$.

In the former case,

$$v = \frac{\alpha\beta}{(p+\alpha)(p+\beta)} . V,$$

for $\alpha\beta = 1/LC$. Assuming partial fractions $\dfrac{Ap}{p+\alpha} + \dfrac{Bp}{p+\beta} + D$, we find

$$v = \left\{1 + \frac{1}{\alpha-\beta}\left(\frac{\beta p}{p+\alpha} - \frac{\alpha p}{p+\beta}\right)\right\} . V$$

$$= V\left\{1 - \left(\frac{\alpha e^{-\beta t} - \beta e^{-\alpha t}}{\alpha - \beta}\right)\right\}.$$

In the latter case,

$$v = \frac{\gamma^2 + \omega^2}{(p+\gamma)^2 + \omega^2} . V$$

$$= \left(1 - \frac{p^2 + 2\gamma p}{(p+\gamma)^2 + \omega^2}\right) . V.$$

Having taken out the '1' term, we need no partial fractions. The standard forms give us

$$v = V\left\{1 - e^{-\gamma t}\left(\cos \omega t + \frac{\gamma}{\omega} \sin \omega t\right)\right\}.$$

In the intermediate case where $R^2 = 4L/C$,

$$v = \frac{\alpha^2}{(p+\alpha)^2} . V$$

$$= \left(1 - \frac{p^2 + 2\alpha p}{(p+\alpha)^2}\right) . V.$$

Here the appropriate partial fractions are $\dfrac{Ap}{p+\alpha} + \dfrac{Bp}{(p+\alpha)^2}$, which leads to

$$v = \left(1 - \frac{p}{p+\alpha} - \frac{\alpha p}{(p+\alpha)^2}\right) . V$$

$$= V\{1 - e^{-\alpha t}(1 + \alpha t)\}.$$

Thus, for small values of R, the condenser voltage oscillates about its final value, gradually settling down to it. As the 'damping component' R is increased, we reach a 'critical'

value where the oscillations disappear. Typical examples, with damping half critical, critical and twice critical, are illustrated in Fig. 10.

Example 2. What alteration is made in the solution of Example 1, if the condenser has an initial charge to voltage v_0?

The p-equation will now have an added v_0 term, so that it will become

$$(LCp^2+RCp+1)\,v = V+(LCp^2+RCp)\,v_0$$

$$= (V-v_0)+(LCp^2+RCp+1)\,v_0.$$

Therefore $$v = v_0+\frac{1}{p^2+\dfrac{R}{L}p+\dfrac{1}{LC}}\cdot\left(\frac{V-v_0}{LC}\right);$$

so $(v-v_0)$ replaces the 'v' of Example 1, and $(V-v_0)$ replaces V. The voltage changes from v_0 to V according to the same law by which it changed in Example 1 from 0 to V.

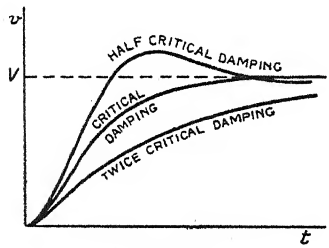

Fig. 10.

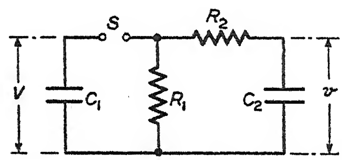

Fig. 11.

Example 3. An impulse generator, which consists essentially of a condenser C_1 charged to a voltage V, applies voltage suddenly (by the sparking over of a spark gap S) to a circuit containing resistances R_1, R_2 and the load capacitance C_2 (previously discharged) (Fig. 11). Find the wave-form of voltage v across the load, given that

$$C_1 = 0{\cdot}02 \text{ microfarad},\quad C_2 = 0{\cdot}001 \text{ microfarad},$$

$$R_1 = 5000 \text{ ohms},\quad R_2 = 1000 \text{ ohms}.$$

The differential equations of this circuit are most conveniently written in terms of v', the voltage across C_1 at any time, as well as

of v; for we know the initial values of both these quantities. If we wrote them in terms of v only, we should need to exercise our minds to think out the initial value of dv/dt in terms of V.

The current through C_2 is $C_2 dv/dt$. Passing through R_2, this causes a voltage drop R_2C_2dv/dt; therefore

$$R_2C_2\frac{dv}{dt}+v = v'.$$

The current drawn from C_1 is $-C_1dv'/dt$. This supplies the current v'/R_1 through R_1, and the current C_2dv/dt through R_2 and C_2. Therefore

$$C_2\frac{dv}{dt}+\frac{v'}{R_1} = -C_1\frac{dv'}{dt}.$$

Rewriting these two equations,

$$R_2C_2\frac{dv}{dt}-v'+v = 0,$$

$$R_1C_1\frac{dv'}{dt}+R_1C_2\frac{dv}{dt}+v' = 0.$$

The p-equations are

$$-v'+(R_2C_2p+1)\,v = 0,$$

$$(R_1C_1p+1)\,v'+R_1C_2pv = R_1C_1pV;$$

for when $t = 0$, $v' = V$ and $v = 0$. Eliminating the unwanted v', we find

$$v = \frac{R_1C_1p}{(R_1C_1p+1)\,(R_2C_2p+1)+R_1C_2p}\,.\,V.$$

The reader will usually find it advantageous to use symbols until this p-expression is reached, and then to insert numbers. When the numerical values are substituted, the units of C and L should be chosen so that the answer may be in terms of a convenient time-unit. Thus:

R in ohms, L in henries, C in farads, lead to t in seconds;

R in ohms, L in millihenries, C in millifarads, lead to t in milliseconds;

R in ohms, L in microhenries, C in microfarads, lead to t in microseconds.

In this problem, the times involved are of the order of the time of discharge of 0·02 microfarad through 5000 ohms—a few score microseconds. We therefore select the microsecond as our time unit.

In numerical terms,

$$v = \frac{100p}{100p^2 + 106p + 1} \cdot V$$

$$= \frac{p}{p^2 + 1{\cdot}06p + 0{\cdot}01} \cdot V.$$

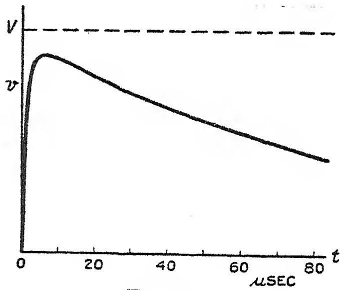

Fig. 12.

The factors of the bottom line prove to be $(p + 1{\cdot}05)\,(p + 0{\cdot}0095)$.

Let $$\frac{p}{(p+1{\cdot}05)\,(p+0{\cdot}0095)} \equiv \frac{Ap}{p+1{\cdot}05} + \frac{Bp}{p+0{\cdot}0095} + C;$$

then $$C = 0, \quad A = -0{\cdot}96, \quad B = +0{\cdot}96.$$

Hence $$v = 0{\cdot}96\left(\frac{p}{p+0{\cdot}0095} - \frac{p}{p+1{\cdot}05}\right) \cdot V$$

$$= 0{\cdot}96V(e^{-0{\cdot}0095t} - e^{-1{\cdot}05t}).$$

The form of this voltage is shown in Fig. 12. It rises steeply to its maximum value, then falls away more gradually to zero.

These three examples adequately illustrate the main propositions of the present chapter. Many more might be added, but it seems best to defer these until after Chapter v, when a great simplification of the method will be given.

CHAPTER IV

THE RESPONSE OF A CIRCUIT TO A SUDDEN SHOCK

4·1. Unit Function.

A differential equation is always the expression of a process of change in a system. For instance, the differential equation of the circuit of Fig. 13, namely

$$RC\frac{dv}{dt}+v = V,$$

might be written

$$\frac{dv}{dt} = \frac{1}{RC}(V-v).$$

This might be verbally expressed thus: 'So long as the condenser voltage v differs from the charging voltage V, it will increase at a rate $(V-v)/RC$.'

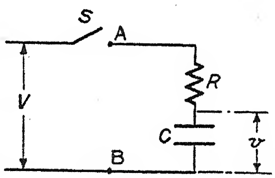

Fig. 13.

Thus the equation focuses our attention on the *process* of change. We shall presently find, however, that the Heaviside method lends itself to a treatment in which the *cause* of the change becomes the centre of interest. We may introduce this treatment by asking ourselves how the process of change in the circuit of Fig. 13 is initiated; the answer is: 'By the closing of the switch S, which causes the voltage across the terminals AB to rise suddenly from zero to V.' This sudden change of voltage gives the circuit a shock, to which its response is expressed by the differential equation.

If V had been, for example, an alternating voltage, the shock to the circuit would have been different, and of course its response also. However, we select one form of shock as our 'standard shock', and build up our theory from it; just as alternating current theory is built upon the sine wave, and waves of more irregular shape are treated as a combination of sine components. The standard shock is a sudden rise (of voltage, current, etc.) from zero to 1, taking place (unless otherwise stated) at time $t = 0$. Before, the quantity is supposed to have the constant

value 0; afterwards, the constant value 1 (Fig. 14). This standard shock is called *unit function*, and is commonly written **1**, or sometimes $H(t)$: H suggesting its originator, Heaviside. The latter notation has the advantage that it lends itself to the expression of a unit function whose leap occurs at time t_0, for this may be written $H(t-t_0)$; but the notation **1** is more compact for ordinary purposes, if this refinement is not required.

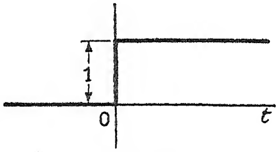

Fig. 14.

A leap in voltage from 0 to V, such as occurs across terminals AB of Fig. 13, will therefore be written $V \times \mathbf{1}$, or $V.\mathbf{1}$. The closing of a switch in a D.C. circuit is in fact one of the commonest ways of applying unit function *voltage* to a circuit, while the opening of such a switch frequently causes a *current* impulse of unit function form. These points will be clarified later—they are mentioned here to show the fundamental importance of the unit function.

4·2. Operations upon Unit Function.

Suppose that a circuit is shocked by unit function voltage $V.\mathbf{1}$, and that the voltage v in some portion of the circuit is obtained in the form

$$v = F(p).(V.\mathbf{1}).$$

We already know how to interpret $F(p)$ operating on V; what if $F(p)$ operates on $V.\mathbf{1}$?

The answer to this question takes us right back to our definition of p, which was

$$\frac{1}{p} = Q = \int_0^t \ldots dt.$$

It will be seen that p only takes account of times subsequent to $t = 0$; therefore the form of the solution is just the same, whether $F(p)$ operates on 1 or on **1**. These two statements crystallize the two ways of looking at the matter.

(1) '$F(p)$ operating on 1 leads to some function of t—say $\Psi(t)$; but this must only be used for positive values of t—negative values are of no significance.'

(2) '$F(p)$ operating on **1** leads to $\Psi(t) \times \mathbf{1}$, which is zero when t is less than 0, and equal to $\Psi(t)$ when t is greater than 0.'

For example, the circuit of Fig. 13 leads to the solution

$$v = V(1-e^{-\alpha t}) \times \mathbf{1}. \text{ (Compare § 2·3, with } v_0 = 0.)$$

This is illustrated in the full curve of Fig. 15. The solution in § 2·3 would have led to the dotted curve, if we had applied it to the era before the closing of the switch; but that would have been foolish, because it is by closing the switch that the whole process is started. The two methods are exactly equivalent if the earlier is seasoned with common sense. Consequently, in the equation

$$F(p).\mathbf{1} = \Psi(t) \times \mathbf{1},$$

the **1** on the right-hand side is nearly always omitted.

Where attention is directed to the *cause* of change, not the *process*, the reader will find that operations on **1** seem more natural than operations on 1; this, however, is just a philosophical distinction, which does not affect the final result.

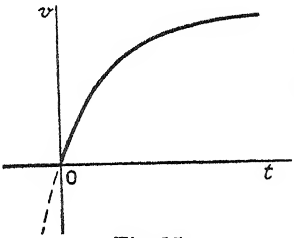

Fig. 15.

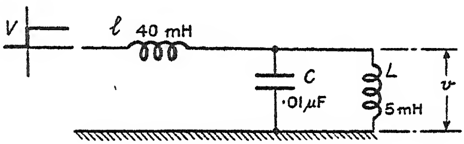

Fig. 16.

4·3. Example. Response of a Circuit to Unit Function Voltage.

A transformer is connected to an overhead line through a cable of capacitance 0·01 microfarad, and a reactor of inductance 40 millihenries; the transformer itself can be regarded as an inductance of 5 millihenries. A lightning stroke causes on the line a very rapid voltage rise of magnitude V (Fig. 16). Find what happens at the transformer terminal.

Let the voltage across the transformer winding be v.

Then the current in C $= C\,dv/dt$.

Therefore the rate of change of current in $C = C\,d^2v/dt^2$.

Also the rate of change of current in L $= v/L$.

Therefore the rate of change of current in l = the sum of these,
$= C\,d^2v/dt^2 + v/L$.

Therefore the voltage across l $= lC\,d^2v/dt^2 + lv/L$,

and the differential equation is

$$lC\frac{d^2v}{dt^2}+\frac{l}{L}v+v = V.1,$$

with the initial conditions $v = 0$, $dv/dt = 0$. The p-equation is

$$\left(p^2+\frac{L+l}{LlC}\right)v = \frac{V}{lC}.1.$$

Substituting $L = 5\times 10^3$ microhenries, $l = 40\times 10^3$ microhenries, $C = 10^{-2}$ microfarad,

$$(p^2+\tfrac{9}{400})v = \frac{V}{400}.1,$$

whence
$$v = \frac{1}{p^2+\frac{9}{400}}\cdot\frac{V}{400}.1$$
$$= \left(1-\frac{p^2}{p^2+(\frac{3}{20})^2}\right)\cdot\frac{V}{9}.1$$
$$= \frac{V}{9}(1-\cos\tfrac{3}{20}t),$$

the multiplying 1 being omitted. This voltage wave is illustrated in Fig. 17.

Though simple, this example contains two features of interest.

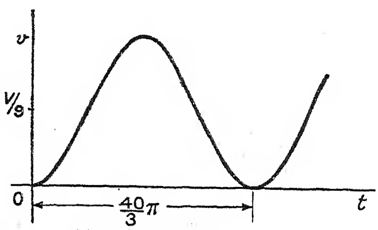

Fig. 17.

(1) The unit function voltage wave is employed as an approximation to a lightning stroke, because this assumption is on the safe side. A real lightning stroke is likely to be less severe in its effects, since

(*a*) Its voltage does not rise with infinite steepness.

(*b*) Its voltage is not sustained indefinitely, but dies away. In this and other problems, unit function supplies a frequent need of the engineer—a simple approximation on the pessimistic side.

(2) The frequency of the oscillation is the same as if L and l were in parallel. They are in fact connected in parallel via the

supposed source of unit function voltage, which must have an effective impedance of zero, because the voltage which it produces is unaffected by the load.

4·4. The Principle of Superposition.

We tend to think of the voltage in a circuit as causing the current, but there is no logical reason why we should; often it is useful to think of the current as causing the voltage. It may even be more convenient to look upon some other quantity—such as the charge on a condenser, or the magnetic flux in an inductance—as the prime cause of the circuit's behaviour. Let us embrace all these possibilities with the non-committal title 'Stimulus', and the symbol U. U will generally be a function of time; thus, the applied stimulus may be a voltage of any wave-form, applied between two chosen points in the circuit. The term 'shock', already used, means 'a stimulus suddenly applied'; the term 'stimulus' includes 'shocks' also.

Suppose the circuit is initially dead; that is, no condenser is charged, no inductance is carrying current. Let a stimulus be applied to the circuit. In each branch, voltage, current, charge or flux will now come into being; but we are probably interested in one property of one branch, so we will fix attention upon it, and call it the 'Response' u. Thus, we may be chiefly interested in the voltage across a particular condenser.

The Principle of Superposition may be stated thus: If the stimulus U applied to a circuit causes a response u in a particular branch, and if the stimulus U' causes in the same branch a response u', then U and U' acting together will produce a response $u+u'$. (Note that there is no need for U and U' to act at the same points in the circuit.) This principle is an immediate consequence of our basic assumption (§ 1·1) that each circuit element is linear. For example,

Current i, in an inductance L, produces a voltage

$$v = L\,di/dt;$$

Current i', in the same inductance, produces a voltage

$$v' = L\,di'/dt;$$

so $(i+i')$ produces a voltage $L(di/dt+di'/dt)$, or $(v+v')$. This would not be true, for example, in a saturated iron-cored inductance.

4·5. Consequences of the Principle of Superposition.

The following consequences scarcely need justification:

(1) If a stimulus is applied to a circuit which is already alive, without the characteristics of the circuit being otherwise altered, the resulting voltages, currents, etc. may be calculated by adding together

(*a*) the voltages, etc., which would have been there if the stimulus had never been applied, and

(*b*) the voltages, etc., which the stimulus would have caused if the circuit had been dead.

The trouble about this is that most ways of applying a stimulus to a circuit (like closing a switch) do alter the circuit characteristics at the same time, and this alteration has to be allowed for. We shall see how to do this in the next chapter. An example of a stimulus which may be applied with only slight alteration in the circuit characteristics is a voltage applied by exciting an alternator, already connected in circuit.

For the rest of this chapter, we shall assume that the circuit is initially dead. The results of any initial 'aliveness' can be dealt with separately.

(2) If the stimulus applied to a circuit consists of a succession of shocks, the response to the stimulus may be obtained by adding together the responses to the shocks.

4·6. Shocks which are not of Unit Function Shape. Duhamel's Integral.

This last statement shows us how to determine the response of a circuit to a shock of any shape, applied at time $t = 0$, when once we know its response to a unit function shock; for the general shock may be treated as a succession of shocks each of unit function form (Fig. 18).

Suppose we know that unit function shock evokes a response u_1, which is a function of t and may therefore be written $u_1(t)$. What happens when we apply a shock of form $U(t)$, at time $t = 0$?

Right at the start, the circuit receives a shock $U(0).\mathbf{1}$, which evokes a response

$$U(0).u_1(t).$$

After that we get the effect of a succession of little jumps of unit function shape. If we desire to know what the circuit is doing at

time t, we must take account of all the little jumps which are engendered up to that time. Let us focus attention on the one which starts at time τ, and let us suppose that a time $\delta\tau$ elapses before the next one. Then the height of that particular jump must be $U'(\tau) \times \delta\tau$, where $U'(\tau)$ denotes the value of dU/dt—the gradient of the U-curve—at that particular instant.

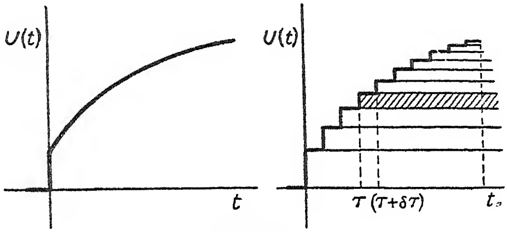

Fig. 18.

Thus at time τ a shock $U'(\tau)\,\delta\tau\,.\,\mathbf{1}$ is applied. When time t comes round, this shock has been effective for a time $(t-\tau)$, and so has caused a response

$$U'(\tau)\,u_1(t-\tau)\,\delta\tau.$$

Consequently the response of the circuit to the whole succession of jumps up to time t is

$$U(0)\,u_1(t) + \sum_{\tau=0}^{\tau=t} U'(\tau)\,u_1(t-\tau)\,\delta\tau,$$

and proceeding to the limit where $\delta\tau$ becomes indefinitely small, we find that the shock $U(t)$ causes a response $u(t)$, given by

$$u(t) = U(0)\,u_1(t) + \int_0^t U'(\tau)\,u_1(t-\tau)\,d\tau.$$

This is called Duhamel's integral. Notice that, in working out the integral, τ is the variable; t is treated as a constant.

By integrating by parts and by other elementary means, we may prove that Duhamel's integral can be written in the following alternative ways:

(1) $$u(t) = U(0)\,u_1(t) + \int_0^t U'(\tau)\,u_1(t-\tau)\,d\tau.$$

(2) $$u(t) = u_1(0)\,U(t) + \int_0^t u_1'(\tau)\,U(t-\tau)\,d\tau.$$

(3) $$u(t) = U(0)\,u_1(t) + \int_0^t u_1(\tau)\,U'(t-\tau)\,d\tau.$$

$$u(t) = u_1(0)\,U(t) + \int_0^t U(\tau)\,u_1'(t-\tau)\,d\tau. \tag{4}$$

$$u(t) = \frac{d}{dt}\left\{\int_0^t U(\tau)\,u_1(t-\tau)\,d\tau\right\}. \tag{5}$$

$$u(t) = \frac{d}{dt}\left\{\int_0^t u_1(\tau)\,U(t-\tau)\,d\tau\right\}. \tag{6}$$

Comparing (1) with (2), (3) with (4) or (5) with (6), we see that U and u_1 are interchangeable. But the presence of $U(0)$ in the first term is always followed by U' in the integral, while $u_1(0)$ in the first term is followed by u_1' in the integral; to notice this may assist the reader's memory.

4·7. Examples in the Use of Duhamel's Integral.

Example 1. In the circuit of Fig. 16, the lightning stroke, instead of causing a voltage change of unit function form, causes a voltage wave

$$V(e^{-at} - e^{-bt}) \quad \text{(Fig. 19)},$$

where $a = \frac{1}{5}$, $b = \frac{2}{3}$. Find the response voltage across the transformer.

The solution already obtained shows that the response to unit function voltage is given by

$$u_1(t) = k(1 - \cos\omega t),$$

where $k = \frac{1}{9}$, $\omega = \frac{3}{20}$. Also $U(t) = V(e^{-at} - e^{-bt})$. We observe that u_1 will be simplified by differentiation, while U lends itself more kindly than u_1 to having t replaced by $(t-\tau)$. (u_1 would split into a sine and a cosine term, whereas U remains single.) We therefore select the second form of Duhamel's integral, and obtain

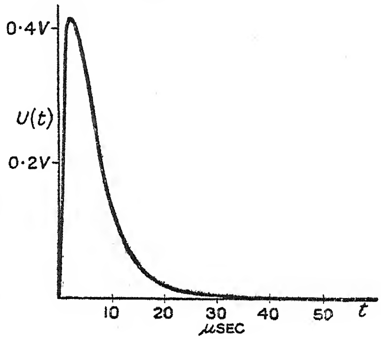

Fig. 19.

$$u(t) = 0 + \int_0^t k\omega \sin\omega\tau \,.\, V(e^{-at}e^{a\tau} - e^{-bt}e^{b\tau})\,d\tau$$

$$= Vk\omega\int_0^t \{e^{-at}(e^{a\tau}\sin\omega\tau) - e^{-bt}(e^{b\tau}\sin\omega\tau)\}\,d\tau.$$

Now $\displaystyle\int_0^t e^{a\tau}\sin\omega\tau\,d\tau = \frac{1}{a^2+\omega^2}\{a\,e^{at}\sin\omega t + \omega(1 - e^{at}\cos\omega t)\}.$

Therefore

$$u(t) = Vk\omega\left(\frac{a\sin\omega t-\omega\cos\omega t+\omega e^{-at}}{a^2+\omega^2}-\frac{b\sin\omega t-\omega\cos\omega t+\omega e^{-bt}}{b^2+\omega^2}\right)$$

$$= Vk\omega\left\{\frac{b-a}{\sqrt{[(a^2+\omega^2)(b^2+\omega^2)]}}\sin(\omega t-\theta)+\frac{\omega}{a^2+\omega^2}e^{-at}-\frac{\omega}{b^2+\omega^2}e^{-bt}\right\},$$

where $\tan\theta = \omega(a+b)/(ab-\omega^2)$.

With numerical values inserted, this becomes

$$u(t) = V\left\{\tfrac{28}{615}\sin\left(\tfrac{3}{20}t-0{\cdot}865\right)+\tfrac{1}{25}e^{-t/5}-\tfrac{9}{1681}e^{-2t/3}\right\}.$$

The form of this voltage is shown in Fig. 20.

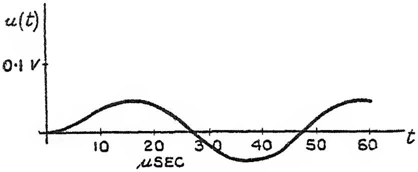

Fig. 20.

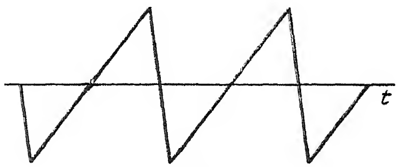

Fig. 21.

Example 2. Duhamel's integral deals very neatly with wave shapes which, though capable of representation by mathematical functions, have points of discontinuity where the function must be changed. Examples of such waves are the saw-toothed voltage forms employed in cathode-ray tube sweep circuits (Fig. 21), and

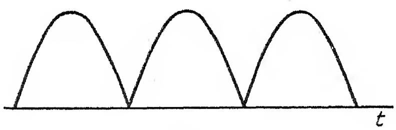

Fig. 22.

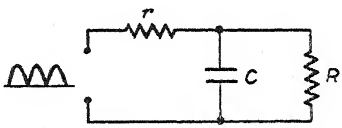

Fig. 23.

the voltage forms produced by rectifiers (Fig. 22). The following is an instance of the latter type.

The simple smoothing circuit (r, C) (Fig. 23) is supplied from a mechanical rectifier which generates half-waves of form $V\sin\Omega t$, and is loaded with a resistance R. Find (in the steady state) the wave-form of voltage across the load.

(The *mechanical* rectifier is specified so as to avoid the added difficulty of dealing with a rectifier like a thermionic valve, which cannot pass a reverse current.)

If v is the voltage across the condenser, the current in C is $C\,dv/dt$, and the current in R is v/R. Therefore the voltage across r is

$$r\left(C\frac{dv}{dt}+\frac{v}{R}\right),$$

and if unit function voltage is applied to the circuit, the differential equation is

$$rC\frac{dv}{dt}+\left(1+\frac{r}{R}\right)v = 1.$$

The p-equation is $rC(p+\alpha)v = 1$, where $\alpha = \dfrac{R+r}{CRr}$. The solution to this is

$$v = \frac{1-e^{-\alpha t}}{\alpha rC},$$

and this is the function which we called $u_1(t)$.

We now deduce from this the effect of shocking the circuit with a single half-sine-wave $V\sin\Omega t$ (Fig. 24), which, since it only lasts from $t = 0$ to π/Ω, should be written

$$U(t) = \Big[V\sin\Omega t\Big]_0^{\pi/\Omega}.$$

We will calculate the response for times greater than π/Ω—that is, for times after the half-sine-wave has come and gone.

We note that $u_1'(t)$ is simpler than $u_1(t)$, and also that $u_1'(t)$ lends itself better than $U(t)$ to having t replaced by $(t-\tau)$. We therefore select the fourth form of Duhamel's integral, and obtain

$$\begin{aligned} v\,(\text{or } u(t)) &= 0+\int_0^t\Big[V\sin\Omega\tau\Big]_0^{\pi/\Omega}\frac{e^{-\alpha(t-\tau)}}{rC}\,d\tau \\ &= \frac{Ve^{-\alpha t}}{rC}\int_0^{\pi/\Omega}e^{\alpha\tau}\sin\Omega\tau\,d\tau, \end{aligned}$$

since the period from time π/Ω to t contributes nothing to the integral,

$$\begin{aligned} &= \frac{Ve^{-\alpha t}}{rC}\left[\frac{e^{\alpha\tau}}{\alpha^2+\Omega^2}(\alpha\sin\Omega\tau-\Omega\cos\Omega\tau)\right]_0^{\pi/\Omega} \\ &= \frac{V\Omega\,(1+e^{\alpha\pi/\Omega})}{rC\,(\alpha^2+\Omega^2)}\,e^{-\alpha t}. \end{aligned}$$

If we measure t from the end of the half-cycle, we get

$$v = \frac{V\Omega(1+e^{-\alpha\pi/\Omega})}{rC(\alpha^2+\Omega^2)}\,e^{-\alpha t}.$$

The contribution of the previous half-cycle is got by increasing t by π/Ω; so it is

$$v = \frac{V\Omega(1+e^{-\alpha\pi/\Omega})}{rC(\alpha^2+\Omega^2)}e^{-\alpha(t+\pi/\Omega)};$$

the one before that contributes

$$v = \frac{V\Omega(1+e^{-\alpha\pi/\Omega})}{rC(\alpha^2+\Omega^2)}e^{-\alpha(t+2\pi/\Omega)},$$

and so on. Therefore all half-cycles before the present one contribute

$$\begin{aligned} v &= \frac{V\Omega(1+e^{-\alpha\pi/\Omega})}{rC(\alpha^2+\Omega^2)}e^{-\alpha t}(1+e^{-\alpha\pi/\Omega}+e^{-2\alpha\pi/\Omega}+\ldots) \\ &= \frac{V\Omega}{rC(\alpha^2+\Omega^2)}\frac{1+e^{-\alpha\pi/\Omega}}{1-e^{-\alpha\pi/\Omega}}e^{-\alpha t}, \end{aligned}$$

provided t is taken as being less than π/Ω.

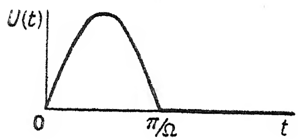

Fig. 24.

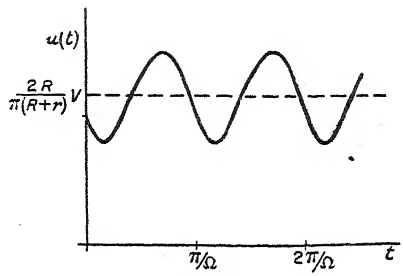

Fig. 25.

Finally, the present half-cycle contributes

$$\begin{aligned} v &= \frac{Ve^{-\alpha t}}{rC}\int_0^t e^{\alpha\tau}\sin\Omega\tau\,d\tau \\ &= \frac{Ve^{-\alpha t}}{rC}\left[\frac{e^{\alpha\tau}}{\alpha^2+\Omega^2}(\alpha\sin\Omega\tau-\Omega\cos\Omega\tau)\right]_0^t \\ &= \frac{V}{rC(\alpha^2+\Omega^2)}(\alpha\sin\Omega t-\Omega\cos\Omega t+\Omega e^{-\alpha t}). \end{aligned}$$

Therefore the total effect of all the half-cycles is

$$v = \frac{V\Omega}{rC(\alpha^2+\Omega^2)}\left(\frac{\alpha}{\Omega}\sin\Omega t-\cos\Omega t+\frac{2}{1-e^{-\alpha\pi/\Omega}}e^{-\alpha t}\right).$$

The form of this voltage is shown in Fig. 25. This method was suggested to the writer by his colleague, Mr D. J. Mynall.

The alternative method of solving this problem would be as follows:

(*a*) Assume a value v_0 for the voltage at the beginning of the cycle.

(*b*) From the differential equation of the circuit, write down the p-equation utilizing this initial value v_0.

(*c*) Solve the p-equation, and from the solution find the value of v at time π/Ω.

(*d*) Equate the initial and final values of v, thereby marking the fact that the circuit is in a steady condition. This equation will give the unknown v_0.

4·8. Note on the Application of Duhamel's Integral to the Aftermath of a Pulse.

A shock which terminates after a finite time, like the half-sine-wave of Fig. 24, may conveniently be termed a 'pulse'. This particular pulse is made simpler to deal with by having no 'step' either at beginning or end; if it had terminated with a step, like the pulse shown in Fig. 26, this would have to be allowed for, thus:

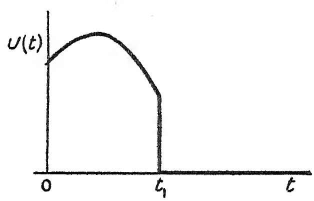

Fig. 26.

The initial step contributes $U(0)\,u_1(t)$ to the integral.

The intermediate part contributes $\int_0^{t_1} U'(\tau)\,u_1(t-\tau)\,d\tau$.

The final step contributes $-U(t_1)\,u_1(t-t_1)$.

The sum of these three may readily be transformed into the single expression

$$\int_0^{t_1} U(\tau)\,u_1'(t-\tau)\,d\tau,$$

which resembles the fourth form of Duhamel's integral, except that the initial term is missing. This is the best form of the integral to use, when calculating the after-effects of a pulse upon a circuit.

4·9. Comparison of Methods for dealing with Shocks which are not of Unit Function Shape.

The reader will probably have recognized that, when the differential equation of a circuit is written down in some such form as

$$a\frac{d^2u}{dt^2}+b\frac{du}{dt}+cu=f(t),$$

the $f(t)$ on the right-hand side is closely related to the stimulus which is applied to the circuit. (See, for instance, Example 2 in § 2·8.) Therefore, contrary to the canons of economical teaching, we have learned three methods of dealing with shocks which are not of unit function shape:

(*a*) In Chapters II and III it was suggested that operational expressions like

$$\frac{1}{p+\alpha}f(t)$$

should be obtained, and interpreted by the rule

$$\frac{1}{p+\alpha}f(t)=e^{-\alpha t}\int_0^t e^{\alpha t}f(t)\,dt.$$

(*b*) In Chapter III $f(t)$ was to be written in the form $\phi(p).1$, and combined with the other p-expressions, the whole lot then being split up into partial fractions and interpreted.

(*c*) In Chapter IV we have proposed solving the problem for a unit function shock, and then deducing the effect of the more complex shape by the use of Duhamel's integral.

I suggest that the choice between these should be made in the following way:

(*a*) is really equivalent to (*c*), and may be considered to have been superseded by it, except perhaps for simple problems; (*b*) involves algebraic processes, which are more elementary and less liable to error than the integration involved in (*c*). Therefore,

When the stimulus is a simple continuous function, use (*b*);

When the stimulus is capable of mathematical representation but has discontinuities, or when no mathematical form is known for it, use (*c*). (If the stimulus is given as a curve of no known mathematical form, Duhamel's integral can be used in conjunction with some method of numerical integration; see § 7·4.)

4·10. The 'Hammer-Blow' Shock.

Besides unit function, there is another standard shock which is worth mentioning: a pulse of extremely high value and short duration, like the blow of a hammer. Such shocks are well known in dynamics, under the title of 'impulses'; unfortunately this word is used rather more vaguely in electrical circuit work, so it has seemed best to avoid it here. A dynamical impulse is an infinite force lasting for an infinitesimal time, but having a time-integral which is finite. The corresponding electrical impulse will therefore be an infinite current lasting for an infinitesimal time, but transporting a finite charge; or an infinite voltage lasting for

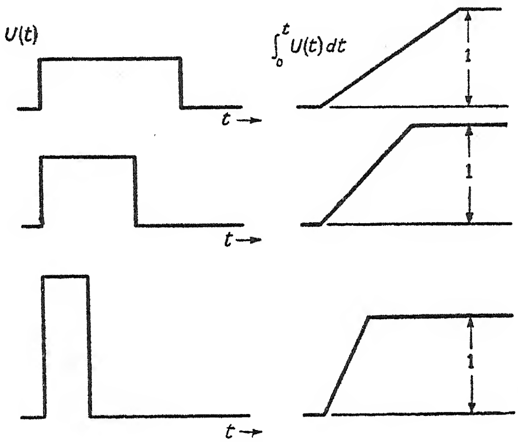

Fig. 27.

an infinitesimal time, but associated with a finite change of magnetic flux. Infinite voltages and currents do not occur in nature, yet they may form a convenient approximation to the truth; for if the duration of a stimulus is small compared with the natural periods of a circuit, there is little error in treating it as a 'hammer-blow'.

Fig. 27 indicates how to find the operational form of such a shock. The pulses $U(t)$ all enclose an area 1; the integral curves shown in the right-hand column all rise to a height 1. As the pulses get shorter, the fronts of the integral curves get steeper; plainly, when the height becomes infinite,

$$\int_0^t U(t)\,dt = 1,$$

or
$$\frac{1}{p}U(t) = \mathbf{1},$$
or
$$U(t) = p.\mathbf{1}.$$
This is the operational form of the standard 'hammer-blow' shock.

4·11. Example. 'Hammer-Blow' Shock.

The circuit of Fig. 16 is struck by a lightning stroke, which causes a steep voltage rise V; but after 1 microsecond this is brought equally abruptly back to zero by an insulator flashover. Find the voltage at the transformer terminal.

The natural period of the circuit is $40\pi/3$, or about 42, microseconds, which is long compared with the duration of the pulse; so we treat the pulse as a 'hammer-blow' of the same area, namely V volt-microseconds.

The differential equation is now
$$lC\frac{d^2v}{dt^2}+\frac{l}{L}v+v = Vp.\mathbf{1},$$
so the p-equation is
$$\left(p^2+\frac{L+l}{LlC}\right)v = \frac{Vp}{lC}.\mathbf{1}.$$
Substituting the values of L, l and C, we get
$$v = \frac{p}{p^2+(\frac{9}{400})}.\frac{V}{400}.\mathbf{1}$$
$$= \frac{\frac{3}{20}p}{p^2+(\frac{3}{20})^2}.\frac{V}{60}.\mathbf{1}$$
$$= \frac{V}{60}\sin\tfrac{3}{20}t.$$

Comparing this with the expression obtained for unit function shock, we see that the maximum amplitude of v is here only $V/60$ instead of $2V/9$. Of course the 'hammer-blow' shock cannot give just the same wave-form as the 'chopped unit function' for the first microsecond; the latter must start in the mode shown in Fig. 17, for how is the transformer to know that the voltage is going to be chopped off, until the chopping happens? But the expression just found is a very good approximation to the subsequent oscillation of v.

CHAPTER V

IMPEDANCE AND ADMITTANCE OPERATORS

5·1. The Analogy between Alternating and Transient Conditions.

When an alternating voltage V is applied to a circuit, whose admittance is known to be
$$G+jB,$$
the components of the current taken by the circuit are given by
$$(G+jB)\times V.$$
Furthermore, if the circuit has several branches, the current in a branch n will be proportional to V, and may therefore be written
$$(G_n+jB_n)\times V,$$
where (G_n+jB_n) is another admittance associated with the circuit, sometimes called the 'transfer admittance' of the branch n. Thus, in Fig. 28, the admittance of the whole circuit is
$$\frac{Cj\omega}{rCj\omega+1}+\frac{1}{Lj\omega+R},$$
while the transfer admittance of the condenser branch is
$$\frac{Cj\omega}{rCj\omega+1}.$$

Now suppose that V, instead of being an alternating voltage, is a voltage shock applied to the terminals of the circuit, which was previously dead. From the preceding chapters, we know that the current which flows into the circuit can be obtained by operating upon V with some p-expression, while the current in any particular branch will be obtained by operating upon V with some other p-expression. In this circuit, for instance,

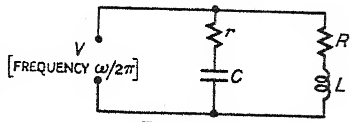

Fig. 28.

$$\text{Current}=\left(\frac{Cp}{rCp+1}+\frac{1}{Lp+R}\right).\{V(t).\mathbf{1}\},$$
and
$$\text{Current in condenser}=\frac{Cp}{rCp+1}.\{V(t).\mathbf{1}\}.$$

V has been written $V(t)$, to remind us that it may have any wave-form.

There is evidently a close analogy between the A.C. and the transient expressions. The currents are obtained from the voltage function $V(t).\mathbf{1}$, by operating upon it with p-expressions known as *admittance operators*. In exactly the same way, the voltages associated with an applied current shock $I(t).\mathbf{1}$ are obtained by operating upon it with *impedance operators*, corresponding to the A.C. equation

$$V = (R+jX) \times I.$$

Moreover, the voltages associated with a voltage shock are obtained by using operators which are really the ratio of two impedance operators, just as in Fig. 28,

$$\text{Voltage across } L = \frac{Lj\omega}{Lj\omega + R} \times V.$$

This little example will have suggested to the reader a fact full of significance: namely, that the impedance and admittance operators have just the same form as the A.C. impedances and admittances, except that $j\omega$ is replaced by p. That this must be so under certain conditions, we shall now proceed to prove.

5·2. Impedance and Admittance Operators in Circuits Initially Dead.

We shall first consider circuits which are dead, up to the instant of application of the shock to which the response is desired; by this, we mean that no inductance is carrying current, no condenser is charged. Another way of saying this would be to state that the circuit has, initially, no stored energy.

At $t = 0$ the shock begins. On each condenser C, the ensuing voltage and current are connected by the relationship

$$i = C\frac{dv}{dt},$$

if the positive direction of current is taken to be that associated with an increase in voltage. Operate on both sides of this equation with the operator

$$\frac{1}{p} \equiv \int_0^t dt.$$

We get

$$\frac{1}{p}i = Cv,$$

since $v = 0$ when $t = 0$. In other words,

$$v = \frac{1}{Cp} i.$$

On each inductance L, the voltage and current are connected by

$$v = L\frac{di}{dt},$$

if the positive direction of voltage is taken to be that associated with an increase in current. Therefore

$$\frac{1}{p} v = Li,$$

since $i = 0$ when $t = 0$. In other words (algebraically),

$$v = Lpi.$$

Exactly similar reasoning applies to each mutual inductance M; here we shall get

$$v_2 = Mpi_1, \quad v_1 = Mpi_2,$$

where 1 and 2 of course denote 'primary' and 'secondary'.

We may therefore formulate the following rules for finding the response when a dead circuit is activated by a shock.

(1) Write down the relationship between the required response u and the known shock $U(t).\mathbf{1}$, treating the circuit like an A.C. circuit in which each condenser has impedance $1/Cp$ and each inductance has impedance Lp or Mp.

(2) Interpret the resulting expression, whose form will be

$$u = F(p).\{U(t).\mathbf{1}\},$$

by carrying out the operations denoted by $F(p)$.

5·3. Examples. Impedance Operators in Circuits Initially Dead.

Example 1. The circuit of Fig. 29 is dead until the switch S is closed, connecting it to a D.C. supply of voltage V. Find the voltage v across the condenser.

The rule just given leads to the result

$$\frac{v}{V} = \frac{1/Cp}{Lp+R+(1/Cp)},$$

or
$$v = \frac{1}{LCp^2+RCp+1}.(V.1).$$

The rest of the solution is just the same as is given in § 3·11, Example 1; but see how much more quickly the p-equation has been obtained!

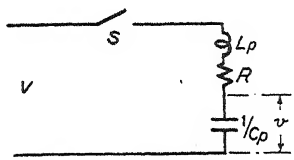

Fig. 29.

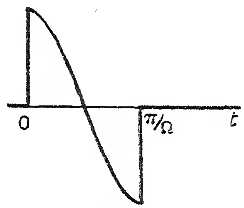

Fig. 30.

Example 2. In a certain oscillograph circuit it is necessary to step down a voltage of half-cosine-wave form (Fig. 30) by a condenser potential divider C_0, C. When an inductance L is connected in parallel with C (Fig. 31), it is found that the stepped-down voltage is distorted in the manner shown in Fig. 32. The second positive peak is undesirable. Account for this peak, and suggest a cure.

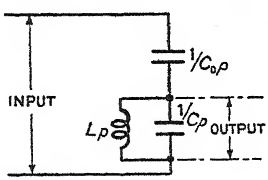

Fig. 31.

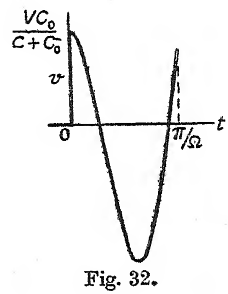

Fig. 32.

We take the input voltage as $(V\cos\Omega t).\mathbf{1}$, up to the time $t = \pi/\Omega$. The output voltage will be v, as usual.

The operational impedance of the circuit of C in parallel with L is

$$\frac{1}{Cp+(1/Lp)}, \quad = \frac{Lp}{LCp^2+1}.$$

Therefore
$$v = \frac{\dfrac{Lp}{LCp^2+1}}{\dfrac{1}{C_0 p}+\dfrac{Lp}{LCp^2+1}}.\{(V\cos\Omega t).1\}$$
$$= \frac{LC_0p^2}{L(C+C_0)p^2+1}.\{(V\cos\Omega t).1\}.$$

We shall interpret this by writing $(\cos\Omega t).1$ in its operational form, $\dfrac{p^2}{p^2+\Omega^2}.1$. Moreover, let $\dfrac{1}{L(C+C_0)} = \omega^2$, so that $\omega/2\pi$ is the natural frequency of the circuit. Then

$$v = \frac{VC_0}{C+C_0}.\frac{p^4}{(p^2+\omega^2)(p^2+\Omega^2)}.1$$
$$= \frac{VC_0}{(C+C_0)(\Omega^2-\omega^2)}.\left(\frac{\Omega^2p^2}{p^2+\Omega^2}-\frac{\omega^2p^2}{p^2+\omega^2}\right).1$$
$$= \frac{VC_0}{C+C_0}\left(\frac{\Omega^2\cos\Omega t-\omega^2\cos\omega t}{\Omega^2-\omega^2}\right).$$

During the interval from $t = 0$ to π/Ω, $\cos\Omega t$ diminishes from $+1$ to -1. If ω is greater than Ω, $\cos\omega t$ will diminish from $+1$ to -1 and then increase again, so that it is quite possible for the whole bracketed expression to be positive in the neighbourhood of $t = \pi/\Omega$. For example, Fig. 32 has been drawn with $\omega = 3\Omega/2$. On the other hand, if ω is less than Ω, the bracketed expression cannot become positive again; therefore a remedy for the second positive peak is to make ω less than Ω—that is, the natural frequency of the circuit should be lower than the frequency of the applied voltage.

5·4. Shock due to Closing or Opening a Switch.

Probably most transient conditions in engineering circuits are brought about by closing or opening a switch, or by some equivalent operation. Closing or opening a switch has two effects upon a circuit:

(1) It applies a voltage or current shock to the circuit.

(2) By altering the interconnection of the circuit elements, it changes the operational impedance of the circuit.

Consider first the effect of closing a switch. While it is still open, there will be in general a voltage across its terminals which may

vary with time in any manner; we will call this voltage $f(t)$ (Fig. 33). In the various arms of the circuit, there will be voltages v_1 and currents i_1, associated with the 'open' condition of the switch; these may also vary with time in any manner.

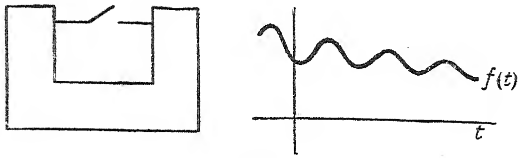

Fig. 33.

Now suppose we have another exactly similar circuit, initially dead as regards both inductances, condensers and generators, but having instead of the switch a generator of zero impedance, which generates a voltage $-f(t).\mathbf{1}$ (Fig. 34). This generator applies to the circuit a voltage shock which leads to response voltages v_2 and currents i_2.

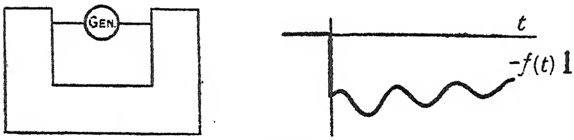

Fig. 34.

Superimpose these two systems of voltage and current. Across the switch we have voltage $f(t)$ before $t = 0$, and zero voltage afterwards—just as if it had been closed. Closing the switch is therefore equivalent to applying a voltage shock $-f(t).\mathbf{1}$; and the voltages and currents which result from closing the switch are $v_1 + v_2$, $i_1 + i_2$. In other words:

The changes produced in a circuit by closing a switch are the same as would be produced by injecting across the switch terminals into a dead circuit a voltage equal and opposite to that across the open switch.

In exactly the same way, we may prove that

The changes produced in a circuit by opening a switch are the same as would be produced by injecting across the switch terminals into a dead circuit a current equal and opposite to that through the closed switch.

It should be added that, in calculating the response of the circuit to a voltage or current shock $-f(t).\mathbf{1}$, all voltage generators in the circuit must be treated as being of zero impedance, all current generators of infinite impedance.

5·5. Impedance and Admittance Operators in Circuits not Initially Dead.

The whole basis of impedance operators, as so far described, rests upon the assumption that the circuit is initially dead. This is obviously a grave limitation upon the method. Of course the methods of Chapters II and III gave us a way of treating circuits not initially dead, for in such a p-equation as

$$(ap^2+bp+c)\,u=f(t)+(ap^2+bp)\,u_0+ap\,u_1,$$

the u_0 and u_1 terms represent the initial state. But that involves writing down the differential equation, and jettisoning the pretty simplicity of the impedance operator method. Can we not retain this simplicity, even when the circuit is initially alive?

Broadly speaking, this end can be attained in two ways. One way is to split up the problem into parts which each involve a shock on a dead circuit; the principle of superposition justifies this. The other is a rule of thumb method for writing down the p-equation for any initial conditions.

5·5 (*a*). Superposition Method.

In § 5·4 we saw that the *change* produced in a circuit by closing (or opening) a switch could be calculated by treating the circuit as dead. Adding this change to the conditions which would have existed if the switch had not been closed (or opened), we obtain the actual state which results from the switching operation.

Now in many cases the conditions, which would have existed if the switch had not been operated, are extremely simple. Often, for instance, those conditions are steady—perhaps there may be a condenser which remains fully charged until a switch is closed. In such a case the only real problem is to calculate the change due to the switching; and this may be done by direct application of impedance operators.

Example 1. In the impulse generator circuit of Fig. 35 the condenser C_1 charged to voltage V is suddenly connected to the rest of the circuit by the sparking over of the spark gap S. Find the voltage v across the load condenser C_2.

Before the sparkover of the gap, terminal A is at potential V with reference to terminal B. The sparkover applies a shock $-V.1$ to A relatively to B, or $+V.1$ to B relatively to A. So the changes produced in the circuit of Fig. 35, when S becomes conducting, are the same as the response of the circuit of Fig. 36 (initially dead) to a shock $V.1$.

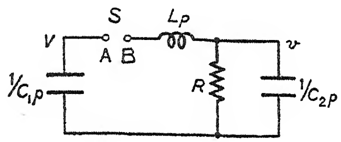

Fig. 35.

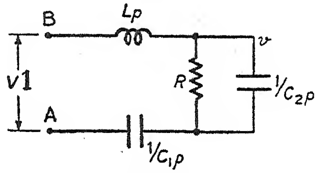

Fig. 36.

The operational impedance of C_2 with R in parallel is

$$\frac{1}{C_2p+(1/R)}, \quad = \frac{R}{RC_2p+1}.$$

Therefore
$$v = \frac{\dfrac{R}{RC_2p+1}}{\dfrac{R}{RC_2p+1}+Lp+\dfrac{1}{C_1p}}.V.1$$

$$= \frac{RC_1p}{(LC_1p^2+1)(RC_2p^2+1)+RC_1p}.V.1.$$

If S had never been closed, v would have been persistently zero; therefore there is nothing to add to this expression for v. The factors of the bottom line in this p-expression can take different forms according to the relative values of R, L, C_1 and C_2, so this is the point at which numbers should be inserted.

To illustrate the necessity, sometimes, for adding the conditions which would have existed if the switch had not been closed, let us calculate the voltage on C_1. From Fig. 36, the change in this voltage is

$$\frac{\dfrac{1}{C_1p}}{\dfrac{R}{RC_2p+1}+Lp+\dfrac{1}{C_1p}}.V.1$$

$$= \frac{RC_2p+1}{(LC_1p^2+1)(RC_2p+1)+RC_1p}.V.1.$$

The polarity of this change is such as to make A the 'negative' terminal, which is opposite to the initial polarity. Therefore, superimposing the change upon the initial conditions, we find

$$\text{Voltage on } C_1 = V - \frac{RC_2p+1}{(LC_1p^2+1)(RC_2p+1)+RC_1p} \cdot V \cdot \mathbf{1}.$$

Example 2. A generator field of resistance R and inductance L is drawing a steady current from a D.C. voltage source V, when the contact S of a Tirrill regulator is opened, inserting into the circuit the resistance r and condenser C (Fig. 37). Find the voltage v which appears across the regulator contacts.

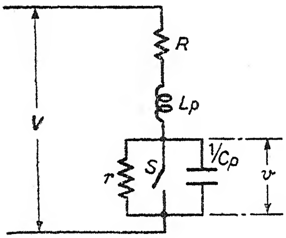

Fig. 37.

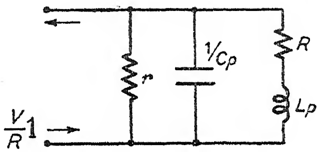

Fig. 38.

Initially, the contacts were passing a steady current V/R. The effect of opening them is to apply a current shock $(V/R).\mathbf{1}$ across them; and, as has been explained already, the voltage source must be treated as having zero impedance. Fig. 38 therefore shows the equivalent problem.

The three current paths are in parallel. Their operational impedance is

$$\frac{1}{Cp+\dfrac{1}{r}+\dfrac{1}{Lp+R}}, \quad = \frac{r(Lp+R)}{rLCp^2+(RrC+L)p+(R+r)}.$$

Therefore the voltage across the contacts is

$$v = \frac{r(Lp+R)}{rLCp^2+(RrC+L)p+(R+r)} \cdot \frac{V}{R} \cdot \mathbf{1}$$

$$= \frac{p+(R/L)}{(p+\alpha)(p+\beta)} \cdot \frac{V}{RC} \cdot \mathbf{1},$$

where $\quad \alpha+\beta = (RrC+L)/rLC, \quad \alpha\beta = (R+r)/rLC.$

(This would be the convenient form to use if the factors of the bottom line were real.)

Taking partial fractions

$$\frac{p+(R/L)}{(p+\alpha)(p+\beta)} \equiv \frac{Ap}{p+\alpha}+\frac{Bp}{p+\beta}+D,$$

we find

$$A=\frac{(R/L)-\alpha}{\alpha(\alpha-\beta)}, \quad B=-\frac{(R/L)-\beta}{\beta(\alpha-\beta)}, \quad D=\frac{R}{L\alpha\beta}=\frac{RrC}{R+r}.$$

Therefore

$$v=\frac{V}{RC}\left\{\frac{(R/L)-\alpha}{\alpha(\alpha-\beta)}e^{-\alpha t}-\frac{(R/L)-\beta}{\beta(\alpha-\beta)}e^{-\beta t}+\frac{RrC}{R+r}\right\},$$

or $$\frac{v}{V}=\frac{r}{R+r}+\frac{1}{LC(\alpha-\beta)}\left\{\left(\frac{1}{\alpha}-\frac{L}{R}\right)e^{-\alpha t}-\left(\frac{1}{\beta}-\frac{L}{R}\right)e^{-\beta t}\right\}.$$

There is nothing to add to this expression, for v would have remained zero if the contacts had not been opened.

5·5 (*b*). Rule of Thumb Method.

Where the state of affairs before the switching is less simple, and in all cases of doubt, it is useful to be able to fall back upon a rule of thumb method for obtaining the p-equation.

Suppose a circuit has n elements, that is, the total number of resistances, inductances and condensers is n, each mutual inductance counting as two. We know all about its behaviour, if we know the voltage v and current i in each element. To find these $2n$ unknowns, we need $2n$ differential equations. n of them are provided by the relation between voltage and current in the individual elements: $v=Ri$, $v=L\,di/dt$, etc. The other n are provided by Kirchhoff's well-known laws of circuit behaviour, namely

(1) The sum of the voltage drops round any closed circuit is equal to the sum of the e.m.f.'s in that circuit.

(2) The sum of the currents arriving at any point is zero.

To write down the p-equations of a circuit under any initial conditions, all we need is to know the p-relation between the voltage and current in each element.

The initial state of the circuit is completely specified if we know the current in every inductance, the primary and secondary currents in every mutual inductance, and the voltage on every

condenser. Suppose we know that an inductance L has initially a current i_0. The voltage across it is given by

$$v = L\frac{di}{dt}.$$

Therefore $$\frac{1}{p}v = L(i-i_0),$$

or (algebraically), $$v = Lp(i-i_0).$$

This is the p-relation between voltage and current in an inductance with initial current i_0. Similarly, in a mutual inductance

$$v_1 = L_1p(i_1-i_{10})+Mp(i_2-i_{20}),$$

$$v_2 = Mp(i_1-i_{10})+L_2p(i_2-i_{20}).$$

And in a condenser, $$i = Cp(v-v_0).$$

We can therefore formulate the following rules for writing down the p-equation under any initial conditions:

(1) Take as separate variables the current in each inductance, the two currents in each mutual inductance, and the voltage on each condenser.

(2) Assuming the above relations between voltage and current in the circuit elements, write down the Kirchhoff equations for voltage and current balance. (The switching operation, or whatever initiates the disturbance, must be regarded as having taken place if the circuit connections are affected thereby.)

(3) Eliminate the unwanted variables, thereby obtaining a p-expression for the one which is required.

This method directs attention to the process rather than to the cause of change, so, as suggested in § 4·2, the symbol **1** does not appear.

Example 1. Let us apply this method to the problem of Fig. 37. The variables shall be the current in L (initial value V/R), and the voltage across C (initial value 0).

After the contacts S have been opened, we have the voltage equation

$$v+Lp\left(i-\frac{V}{R}\right)+Ri = V,$$

since the voltage on L is $Lp(i-i_0)$; or

$$v+(Lp+R)\,i = V+\frac{Lp}{R}V;$$

and also the current equation

$$Cpv+\frac{v}{r} = i,$$

or
$$\left(Cp+\frac{1}{r}\right)v-i = 0.$$

Eliminating i between these equations, we find

$$\left\{(Lp+R)\left(Cp+\frac{1}{r}\right)+1\right\}v = \left(\frac{Lp}{R}+1\right)V,$$

whence
$$v = \frac{r(Lp+R)}{rLCp^2+(RrC+L)\,p+(R+r)}\cdot\frac{V}{R},$$

just as before.

Example 2. In a certain oscillograph circuit, an operation equivalent to closing a switch S connects the circuit shown in Fig. 39 to an alternating voltage source at the instant of voltage maximum. When S is closed, a current i_{20} is still flowing in the secondary circuit, the residue of a previous cycle. Find the changes in secondary current afterwards.

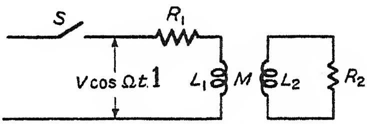

Fig. 39.

By the rules given, the primary voltage of the mutual inductance is

$$L_1pi_1+Mp(i_2-i_{20});$$

the secondary voltage,

$$Mpi_1+L_2p(i_2-i_{20}).$$

Therefore

$$(L_1p+R_1)\,i_1+Mp(i_2-i_{20}) = V\cos\Omega t$$
$$Mpi_1+L_2p(i_2-i_{20})+R_2i_2 = 0;$$

or
$$(L_1p+R_1)\,i_1+Mpi_2 = V\cos\Omega t+Mpi_{20},$$
$$Mpi_1+(L_2p+R_2)\,i_2 = L_2pi_{20}.$$

Therefore

$$\{(L_1p+R_1)\,(L_2p+R_2)-M^2p^2\}\,i_2$$
$$= -MpV\cos\Omega t+\{(L_1L_2-M^2)\,p^2+R_1L_2p\}\,i_{20}.$$

This may be written

$$(L_1L_2-M^2)(p+\alpha)(p+\beta)\,i_2$$
$$= -M\frac{p^3}{p^2+\Omega^2}V+(L_1L_2-M^2)(p^2+\gamma p)\,i_{20},$$

where

$$\alpha+\beta = \frac{R_1L_2+R_2L_1}{L_1L_2-M^2}, \quad \alpha\beta = \frac{R_1R_2}{L_1L_2-M^2}, \quad \gamma = \frac{R_1L_2}{L_1L_2-M^2}.$$

Hence

$$i_2 = -\frac{M}{L_1L_2-M^2}\frac{p^3}{(p+\alpha)(p+\beta)(p^2+\Omega^2)}\cdot V+\frac{p^2+\gamma p}{(p+\alpha)(p+\beta)}\cdot i_{20}.$$

Each term can be interpreted by splitting into partial fractions, as usual.

The reader may find it instructive to attempt this problem by the superposition method. The voltage, which would have existed across the switch if it had not been closed, is complicated by a voltage induced by the dying current whose instantaneous value was i_{20}. The method just given takes account of all this without effort.

CHAPTER VI

RESONANCE, DAMPING AND STABILITY

6·1. Deductions from the p-Equation.

In the preceding chapters we have bent our energies to finding a complete expression for the response of the circuit as a function of time. But in many problems we may be content with less than this; we may wish to know merely the kind of response which the circuit will make, or what changes must be made in the circuit to bring about a response of a more desirable kind. Such information as this can often be deduced directly from the p-equation, without going through the ensuing process of interpretation.

When we obtain a p-equation such as

$$u = \frac{\gamma p}{(p+\alpha)(p+\beta)} . 1,$$

the type of solution is controlled by the factors of the bottom line of the p-expression; for these determine the partial fractions into which this p-expression shall be split up. In the case just quoted, these are

$$u = \frac{\gamma}{\alpha-\beta}\left(\frac{p}{p+\beta} - \frac{p}{p+\alpha}\right) . 1,$$

leading to $$u = \frac{\gamma}{\alpha-\beta}(e^{-\beta t} - e^{-\alpha t}).$$

Differences in the top line lead to different solutions of the same type; thus

$$u = \frac{\gamma p^2}{(p+\alpha)(p+\beta)} . 1$$

gives $$u = \frac{\gamma}{\alpha-\beta}(\alpha e^{-\alpha t} - \beta e^{-\beta t}),$$

a different combination of the same exponentials.

If the p-equation were

$$(ap^2+bp+c)\,u = f(t) + (ap^2+bp)\,u_0 + apu_1$$

it would be necessary to express $f(t)$ in the form $\phi(p) . 1$, before examining the p-expressions. That part of the solution which

depends upon u_0 and u_1 will have its type controlled by the factors of (ap^2+bp+c). The part depending upon $f(t)$ will in general contain these terms too, but it will also contain other terms corresponding to the factors of the bottom line of $\phi(p)$.

To determine the type of response u which the circuit will make, it is therefore only necessary to express u in the form

$$u = F(p).1,$$

and examine the bottom line of the fraction $F(p)$.

6·2. Resonance.

A simple example is afforded by the phenomenon of Resonance, which occurs when an alternating stimulus is applied to a circuit, one of whose natural frequencies is equal to the frequency of the stimulus. An abnormally large response is apt to occur, as the reader may recall from steady-state A.C. theory. The following example illustrates what happens under transient conditions.

6·3. Example. Resonance.

A voltage $(V \sin \Omega t).1$ is applied to an inductance L and condenser C in series, and $\Omega^2 = 1/LC$ (Fig. 40). Find the current.

The operational admittance of the circuit is

$$\frac{1}{Lp+(1/Cp)}, = \frac{Cp}{LCp^2+1}$$

$$= \frac{1}{L}\frac{p}{p^2+\Omega^2}.$$

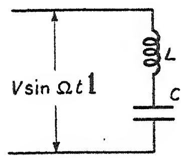

Fig. 40.

Therefore the current is given by

$$i = \frac{1}{L}\frac{p}{p^2+\Omega^2}.\{(V \sin \Omega t).1\}$$

$$= \frac{1}{L}\frac{p}{p^2+\Omega^2}.\frac{V\Omega p}{p^2+\Omega^2}.1$$

$$= \frac{\Omega V}{L}\frac{p^2}{(p^2+\Omega^2)^2}.1$$

$$= \frac{V}{2L}t \sin \Omega t.$$

The current therefore increases without limit, after the fashion shown in Fig. 41.

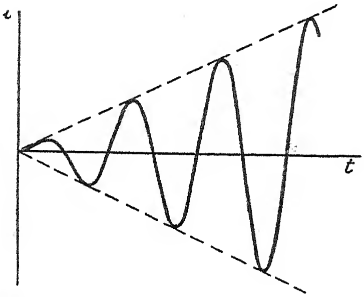

Fig. 41.

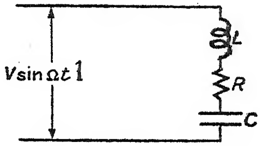

Fig. 42.

6·4. Example. Resonance with Losses.

In practice the current will not increase without limit, because losses ignored in § 6·3 limit its growth. Thus, if the inductance L has associated with it a resistance R (Fig. 42), the operational admittance becomes

$$\frac{1}{L}\frac{p}{p^2+(R/L)p+(1/LC)}, \quad = \frac{1}{L}\frac{p}{p^2+2\alpha p+\Omega^2},$$

where $\alpha = R/2L$; α is supposed to be small compared with Ω.

The current is now

$$i = \frac{\Omega V}{L}\frac{p^2}{(p^2+\Omega^2)(p^2+2\alpha p+\Omega^2)}\cdot 1$$

$$= \frac{\Omega V}{L}\frac{1}{2\alpha}\left(\frac{p}{p^2+\Omega^2}-\frac{p}{p^2+2\alpha p+\Omega^2}\right)\cdot 1$$

$$= \frac{V}{R}\left(\sin\Omega t-\frac{\Omega}{\omega}e^{-\alpha t}\sin\omega t\right),$$

where $\omega^2 = \Omega^2-\alpha^2$. The form of this is shown in Fig. 43. It tends to the final value

$$\frac{V}{R}\sin\Omega t,$$

instead of to infinity.

The occurrence of resonance is signalized by the appearance of a squared factor of form $(p^2+\Omega^2)^2$, in the bottom line of the fraction $F(p)$; or, when losses are taken into account, by a product of form $(p^2+\Omega^2)(p^2+2\alpha p+\Omega^2)$.

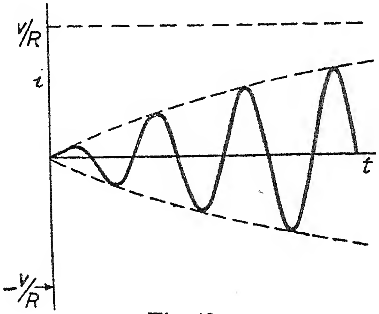

Fig. 43.

6·5. Damping.

The study of the p-expression $F(p)$ in any problem enables us to determine a matter of great practical importance—the degree of damping which attends the oscillations of the circuit. Look back for a moment at the circuit of Fig. 9, whose different types of response are shown in Fig. 10. If the resistance R could be made zero, the voltage shock V would lead to a response v in the form of an oscillation persisting indefinitely. As R is made greater, the amplitude of the oscillation dies away more and more rapidly, until the 'critical state' is reached in which the oscillations disappear altogether (Fig. 10). If R is made greater still, the response has no oscillatory tendency, but is represented by a combination of exponentials. This is typical of the behaviour of damped oscillatory circuits; a progressive change in the circuit resistances is associated with a progressive change in the degree of damping.

It has already been observed that the clue to the behaviour of any particular circuit is to be found by examining the bottom line of the fraction $F(p)$; this will have the form

$$ap^n+bp^{n-1}+\ldots+lp+m,$$

and by finding the factors of this we can determine what kind of terms will occur in the solution, and whether oscillations are present or not. For every real factor $(p+\alpha)$, a term $Ae^{-\alpha t}$ will occur; for every quadratic factor $(p^2+\beta p+\gamma)$, where $\beta^2<4\gamma$,

there will be terms like $e^{-\beta t/2}(B\cos\omega t + C\sin\omega t)$. Thus we can quickly determine the type of response which will be made by a circuit whose components are numerically known.

But this is not enough. Often we desire to know, not only what the circuit will do, but how to make it do something else. It oscillates, and we want to stop it oscillating; or it refuses to oscillate, and we want to know how to encourage it to do so. Some of the values of the circuit elements are at our disposal; how must we dispose them to make the circuit do what we want?

To answer this question, we need a general criterion by which we can determine whether the factors of the bottom line of $F(p)$ are real or not. Writing this criterion in terms of the disposable circuit constants, we can find what relation must be satisfied by those constants in order that oscillations may be possible or impossible. In Appendix V, where the properties of expressions like

$$ap^n + bp^{n-1} + \ldots + lp + m$$

are briefly discussed, the reader will find these necessary criteria for expressions of the second, third and fourth degree; the use of them will be illustrated by two examples, one of which is worked out very fully in order to illustrate certain features of this type of problem.

6·6. Examples. Damping.

Example 1. A circuit consisting of L and C in parallel is shocked by voltage of unit function form, applied through resistance R (Fig. 44). Find what condition R must satisfy, in order that the voltage v across L and C may oscillate.

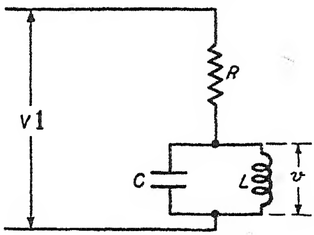

Fig. 44.

The operational impedance of L and C in parallel is

$$\frac{1}{Cp + (1/Lp)}, \quad = \frac{Lp}{LCp^2 + 1}.$$

Therefore
$$\frac{v}{V} = \frac{\dfrac{Lp}{LCp^2+1}}{R + \dfrac{Lp}{LCp^2+1}}.1$$

$$= \frac{Lp}{RLCp^2 + Lp + R}.1.$$

The condition for the factors of the bottom line *not* to be real is given in Appendix V; in this instance, it leads to the result

$$L^2 < 4R^2LC,$$

or

$$R > \frac{1}{2}\sqrt{\frac{L}{C}},$$

which is the condition required.

Example 2. A transformer is connected to a supply by way of a switch S; but between them is an overhead line of inductance l. Under certain conditions of imperfect operation of S, a large steep-fronted voltage shock $V.1$ becomes applied to the terminals A, B; and by oscillation of the line inductance l with the transformer capacitance C, the transformer experiences an oscillatory voltage of crest value $2V$. (The transformer inductance L', and the line capacitance c, exert a negligible influence on this effect.)

It is proposed to prevent this undesirable doubling by inserting a series reactor L damped by a resistance R (Fig. 45). Clearly such an insertion cannot be effective if L is made too small.

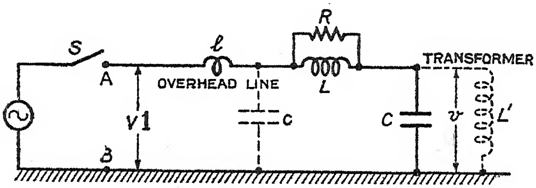

Fig. 45.

(1) How large must L be, if critical damping is to be possible?

(2) If L can be made as large as this, what value should be given to R?

(3) If L cannot be made as large as this, what value of R will give maximum damping?

(1) By the usual method,

$$\frac{v}{V} = \frac{1/Cp}{(1/Cp) + lp + (RLp/[Lp+R])} \cdot 1$$

$$= \frac{Lp+R}{LlCp^3 + RC(L+l)p^2 + Lp + R} \cdot 1.$$

From Appendix V, the condition for over-critical damping is

$$27(LlC)^2 R^2 - [RC(L+l)]^2 L^2 + 4(LlC) L^3 + 4[RC(L+l)]^3 R - 18(LlC) RC(L+l) LR < 0,$$

which reduces to

$$\{4(L+l)^3 C^2\} R^4 + \{27L^2l^2C - L^2(L+l)^2 C - 18L^2(L+l) lC\} R^2 + \{4L^4l\} < 0.$$

Though at first sight this expression looks ugly, containing as it does so high a power as R^4, it is not so bad as it looks; for it only contains R^4 and R^2, so it may be described as being of the second degree in R^2. If L, as well as l and C, were specified, this inequality would enable us to determine whether any given value of R gave over-critical damping or not.

Under what conditions can a quadratic expression become negative? Think of such an expression as

$$ax^2 + bx + c,$$

where a is positive. The 'roots', or values of x which make this equal to zero, may be real (say f, g), or imaginary (say $h+jk$, $h-jk$). If they are real, the expression has factors

$$a(x-f)(x-g),$$

and this product is negative if, and only if, x lies between f and g. If they are imaginary, the factors are

$$a(x-h-jk)(x-h+jk), \quad \equiv a\{(x-h)^2 + k^2\},$$

and this, being the sum of two squares, can never be negative. Thus a quadratic expression can only be negative if its roots are real and the variable lies between them.

Our condition for over-critical damping can therefore never be satisfied unless the roots of the R^2 expression are real. The condition for this is

$$\{27L^2l^2C - L^2(L+l)^2 C - 18L^2(L+l) lc\}^2 > 4\{4(L+l)^3 C^2\}\{4L^4l\},$$

or
$$\{27l^2 - (L+l)^2 - 18l(L+l)\}^2 > 64l(L+l)^3,$$

or
$$L^4 - 24L^3l + 192L^2l^2 - 512Ll^3 > 0,$$

or
$$L(L-8l)^3 > 0,$$

or
$$L > 8l.$$

This shows how large L must be for critical damping to be possible.

(2) If L is made equal to $8l$, the roots of the R^2 expression are equal; there is then just one value of R which gives critical damping. Substituting $8l$ for L, we find that the R^2 expression reduces to

$$729C^2R^4 - 3456lCR^2 + 4096l^2 = (3^3CR^2 - 2^6l)^2.$$

Critical damping occurs if this is zero; i.e. if

$$R^2 = \frac{2^6}{3^3}\frac{l}{C},$$

or

$$R = 1{\cdot}54\sqrt{\frac{l}{C}}.$$

This answers question (2).

(3) If L is made less than $8l$, critical damping cannot be attained. The expression for v/V can therefore have its bottom line factorized in the following way:

$$\frac{v}{V} = \frac{Lp+R}{LlC(p+\alpha)\{(p+\beta)^2+\omega^2\}}\cdot 1.$$

For damping which is less strong than critical damping will permit damped oscillations (of the form $e^{-\beta t}\cos\omega t$, $e^{-\beta t}\sin\omega t$) to appear, and the corresponding factor in the p-expression is $\{(p+\beta)^2+\omega^2\}$. We want to know how to make the damping as severe as possible—that is, we want β to be as large as possible. Our first task, therefore, must be to find an expression for β in terms of R, L, l and C.

Now

$$LlC(p+\alpha)\{(p+\beta)^2+\omega^2\}$$

was written for

$$LlCp^3 + R(L+l)\,Cp^2 + Lp + R.$$

Therefore, equating the coefficients of p^2, p and 1 on the two sides, we find

$$\alpha + 2\beta = R\left(\frac{L+l}{Ll}\right) = \frac{R}{L_0}, \text{ say;}$$

$$2\alpha\beta + \beta^2 + \omega^2 = \frac{1}{lC},$$

and

$$\alpha(\beta^2+\omega^2) = \frac{R}{LlC}.$$

From these three equations we can eliminate α and ω, so obtaining an equation for β. The third of the equations gives

$$(\beta^2+\omega^2) = \frac{R}{\alpha LlC},$$

whence, substituting in the second, we get rid of ω, and obtain

$$2\alpha\beta+\frac{R}{\alpha LlC} = \frac{1}{lC}.$$

In this may be substituted the value of α given by the first equation, namely $(R/L_0)-2\beta$. The result, as we desired, is an equation involving β only, which will be found to boil down to the following cubic:

$$(2\beta)^3-2\frac{R}{L_0}(2\beta)^2+\left(\frac{R^2}{L_0^2}+\frac{1}{lC}\right)(2\beta)-\frac{R}{l^2C} = 0.$$

It must be constantly borne in mind that L_0, l and C are fixed, whereas we are able to select any value of R that we please. For each selected value of R, the real root of the cubic equation determines a corresponding value of β; the curve relating the two will be something like Fig. 46, for β is zero when $R = 0$ or ∞, but takes positive values when R is in between. What we want to know is the value R_1, which makes β have its maximum value β_1.

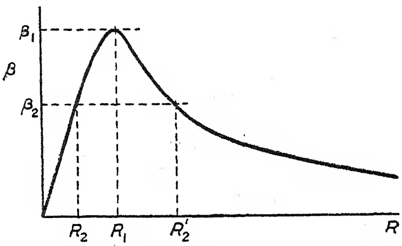

Fig. 46.

The cubic equation for (2β) contains R to the second power; in fact, it might be rewritten

$$\left(\frac{2\beta}{L_0^2}\right)R^2-\left\{\frac{2(2\beta)^2}{L_0}+\frac{1}{l^2C}\right\}R+\left\{(2\beta)^3+\frac{(2\beta)}{lC}\right\} = 0.$$

For any particular value of β, say β_2 (Fig. 46), this equation gives two values of R—R_2 and R_2'. If the two values of R coincide, it can only mean that $\beta = \beta_1$; the condition for this is

$$\left(\frac{8\beta^2}{L_0}+\frac{1}{l^2C}\right)^2 = \frac{8\beta}{L_0^2}\left(8\beta^3+\frac{2\beta}{lC}\right).$$

Therefore β_1 is the solution of the equation last written, which will be found to be

$$\beta_1 = \frac{L}{4l\sqrt{(L+l)\,C}},$$

if we remember that $\quad L_0 = \dfrac{Ll}{L+l}.$

Substituting back in the R-equation, we find

$$R_1 = \frac{L(L+2l)}{2(L+l)\sqrt{(L+l)\,C}}.$$

This is the value of R for maximum damping, if $L < 8l$.

It is interesting to notice that, if $L = 8l$, $R_1 = \dfrac{40}{27}\sqrt{\dfrac{l}{C}}$, or $1{\cdot}48\sqrt{\dfrac{l}{C}}$. Critical damping requires $R = 1{\cdot}54\sqrt{\dfrac{l}{C}}$. Therefore, when $L = 8l$, critical damping is not quite the same thing as maximum damping; though in fact the difference is trifling.

6·7. Stability.

All the problems so far considered have possessed a feature in common—namely, that we could make the maximum value of the response as small as we pleased, if we made the shock small enough. Usually the form of the response has been something like this:

$$\frac{u}{U} = Ae^{-\alpha t} + \ldots + Be^{-\beta t}\cos\omega t + \ldots.$$

The terms on the right-hand side have a perfectly well-defined maximum value, say M; so u has the maximum value MU, which can be made as small as we please if U is sufficiently reduced.

Only once have we seemed to encounter a different state of affairs, and that was in the first resonance example (§ 6·3). There, we found

$$i = \frac{V}{2L}t\sin\Omega t,$$

and it might appear that, no matter how small V might be, we should find that i attained a big value if we waited a very long

time. But this formula was obtained by neglecting the resistance which is inevitably present in every real circuit; and the succeeding example showed that, when this is taken into account, the tendency for i to increase without limit is abolished.

All this seems perfectly natural, until we recall that things are not always like this in other branches of physics. Think, for example, of a pendulum, consisting of a heavy ball on a light pivoted rod. It is theoretically possible for this pendulum to remain at rest in the upward position (Fig. 47); but let it be disturbed from that position by the breadth of a hair, and it will swing right down. An infinitesimally small 'shock' produces here a large 'response'; for this reason, the upward vertical position is said to be 'unstable'. Can an analogous condition be found in an electrical circuit?

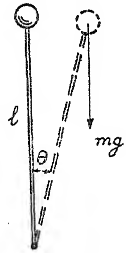

Fig. 47.

Let us write down the equation of motion of the pendulum. We do not know the reaction in the rod, so we resolve the forces on the bob in a direction perpendicular to the rod, when this is inclined at an angle θ; the only force in this direction is the component of the gravitational force mg. Since

Force in any direction = mass × acceleration in that direction,

we have
$$mg\sin\theta = ml\frac{d^2\theta}{dt^2}.$$

Until θ gets too big, we may write θ instead of $\sin\theta$, and obtain
$$\frac{d^2\theta}{dt^2}-\frac{g}{l}\theta = 0.$$

If the initial value of θ was θ_0, we may write the p-equation
$$(p^2-\alpha^2)\,\theta = p^2\theta_0,$$
where
$$\alpha^2 = g/l.$$

Hence, by the usual method,
$$\theta = \tfrac{1}{2}\theta_0(e^{\alpha t}+e^{-\alpha t}).$$

The term which indicates instability is $e^{\alpha t}$; for, no matter how small θ_0 may be, it is possible to make $\frac{1}{2}\theta_0 e^{\alpha t}$ quite big if we make t big enough.

In the electrical circuit problems so far treated, we have never encountered a factor like $e^{\alpha t}$, or $e^{\beta t}\cos\omega t$; only $e^{-\alpha t}$, $e^{-\beta t}\cos\omega t$, etc. Positive powers of e are the signal of instability. They never occur, however, in connection with simple networks of resistance, inductance and capacitance. Where they can occur is in circuits in which the value of one quantity is made to control the value of another—as, for instance, the grid voltage of a valve controls the anode current; the two examples given below will illustrate this. To detect the presence of instability, we need a criterion to determine whether any of the factors of a p-expression have the form $(p-\alpha)$, or $(p^2-\beta p+\gamma)$, where α and β are positive; for such factors will lead to positive powers of e. For expressions up to the fourth degree, the reader will find these criteria in Appendix V.

6·8. Examples. Stability.

Example 1. Fig. 48 shows the circuit of a valve oscillator. If the (L, C) circuit is set in oscillation, an oscillatory voltage is applied to the grid of the valve by way of the mutual inductance M. This causes pulsations in the anode current of the valve, which can—under certain conditions—overcome the natural tendency of the circuit oscillations to die away. It is required to find these conditions.

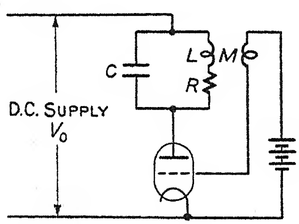

Fig. 48.

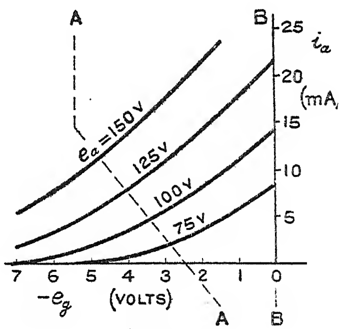

Fig. 49.

The characteristics of a typical valve are shown in Fig. 49. The anode current i_a is dependent jointly upon the anode voltage e_a and the grid voltage e_g, and over a considerable range AB the 'curves' are straight lines, so that we may write

$$i_a = ae_a + be_g + c,$$

where a and b are positive.

Suppose that the circuit is running steadily without oscillation, drawing a direct current i_{a0} from the D.C. supply V_0. The quantities i_{a0}, e_{a0} and e_{g0} must then be related by the equations

$$i_{a0} = ae_{a0} + be_{g0} + c,$$

and $$Ri_{a0} + e_{a0} = V_0.$$

Now suppose that something happens to change the current in (L, R) to the new value $i_{a0}+i$, and the voltage across C to $Ri_{a0}+v$. We can obtain a differential equation for v and i by virtue of the fact that

Voltage across C = Voltage across L and R;

that is $$Ri_{a0} + v = L\frac{di}{dt} + R(i_{a0}+i),$$

or $$v = L\frac{di}{dt} + Ri.$$

We can obtain a second relation from the characteristics of the valve. The steady anode current, anode voltage and grid voltage now change to variable quantities i_a, e_a, e_g, related to the original quantities as follows:

$$\begin{aligned} i_a &= \text{Current in } L + \text{Current in } C \\ &= i_{a0} + i + C\frac{dv}{dt}. \\ e_a &= V_0 - \text{Voltage across } C \\ &= e_{a0} - v. \\ e_g &= e_{g0} + M\frac{di}{dt}. \end{aligned}$$

But $$i_a = ae_a + be_g + c, \quad i_{a0} = ae_{a0} + be_{g0} + c.$$

Therefore $$C\frac{dv}{dt} + i = -av + bM\frac{di}{dt}.$$

We now have two differential equations linking v and i with t. The corresponding p-equations are

$$v - (Lp+R)\,i = \text{terms depending on initial disturbance.}$$

$$(Cp+a)\,v - (bMp-1)\,i = \quad ,, \quad ,, \quad ,, \quad ,,$$

If we are only interested in finding the conditions for continued oscillation, we need not worry about exactly what terms occupy the right-hand side; for these will only appear in the *top* line of the p-expression for v or i, whereas the presence of oscillations is made evident by the *bottom* line. So, eliminating v (for instance), we get

$$\{(Cp+a)(Lp+R)-bMp+1\}\,i = \text{initial terms.}$$

The conditions for sustained oscillation can be found by examining the p-expression in curly brackets, which is

$$LCp^2+(RC+aL-bM)\,p+(1+aR).$$

We need only refer to the criterion of stability given in Appendix V; in this case the 'unstable' condition is the condition to be desired. Having observed that the preliminary condition, $LC>0$, is satisfied, we write down the conditions for the system to be stable, namely

$$(1)\;\; RC+aL-bM>0, \qquad (2)\;\; 1+aR>0,$$

and observe that there will be instability if *one* of these conditions is broken. But condition (2) cannot be broken; so instability is only possible if

$$RC+aL-bM<0.$$

This is not quite enough; for an oscillator is of little use unless it oscillates. The condition for there to be any oscillations at all must also be satisfied, and this is

$$(RC+aL-bM)^2<4LC(1+aR).$$

Failing satisfaction of this second condition, the quantities v, i would simply increase like $e^{\alpha t}$, instead of $e^{\alpha t}\cos\omega t$, etc.

If both these conditions are satisfied, the current will be found to be of the form

$$i = A+e^{\alpha t}(B\cos\omega t+C\sin\omega t),$$

which might seem to betoken an oscillation of ever-increasing amplitude. What actually happens, of course, is that the amplitude increases only until the curved parts of the valve characteristics come into the picture; the assumptions of this discussion then cease to be true, and instead of continuing to increase the oscillation settles down to a steady maximum value.

Example 2. The voltage of a D.C. generator is intended to be held against changes of speed, by a regulator of the form shown in Fig. 50. A coil, energized by the generated voltage, attracts an armature against a constant counter-pull T_0; the armature is coupled to the moving contact of a variable resistance in the field circuit. A momentary increase in generated voltage, due to an increase of speed, should therefore be automatically corrected by an increase in the resistance of the field circuit. 'Armature reaction' in the generator is neglected.

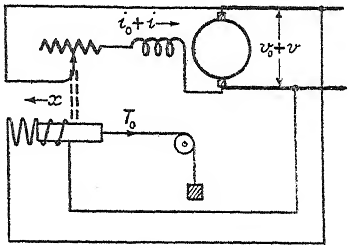

Fig. 50.

The movements of the regulator are damped by an oil dashpot (not shown). It is required to find how much damping must be provided, in order to prevent the system from oscillating persistently.

It will be observed that this system can only be in equilibrium when the generated voltage has a particular value v_0, namely that voltage for which the solenoid pulls the armature with a force T_0. If the speed has its normal value n_0, we will suppose that a field current i_0 suffices to generate this voltage. Then the resistance R_0 in the field circuit must be given by

$$v_0 = R_0 i_0.$$

Now suppose that the speed suddenly changes to n_0+n, where n/n_0 is small, and remains constant at this new value. Everything else starts to change in sympathy. At a subsequent instant, let the generated voltage be v_0+v, and the field current i_0+i; furthermore, let the regulator armature move a distance x from its initial equilibrium position, increasing the field resistance by a proportional amount which we will call ax.

Our first equation will come from the mechanical behaviour of the regulator armature. If the extra voltage v causes an unbalanced force bv, this equation is

$$m\frac{d^2x}{dt^2}+k\frac{dx}{dt} = bv,$$

where m is the mass of the armature, and k is the damping coefficient associated with the dashpot.

Secondly, we have the relation between voltage and current in the field circuit; this is

$$L\frac{di}{dt}+(R_0+ax)(i_0+i) = (v_0+v),$$

where L is the self-inductance of the field. Neglecting the product axi (for both x and i are small), and remembering that $R_0 i_0 = v_0$, we may rewrite this equation in the simpler form

$$L\frac{di}{dt}+R_0 i+ai_0 x = v.$$

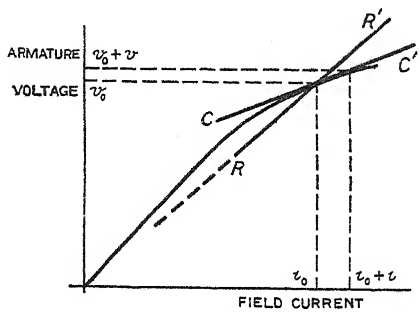

Fig. 51.

The third equation involves the saturation curve of the generator. This will have the form shown in Fig. 51. At constant speed n_0, the increase v in voltage is related to the increase i in field current by the equation

$$v = ci,$$

where c is the gradient of the tangent CC' to the saturation curve; but the speed variation changes this to

$$(v_0+v) = (v_0+ci)\left(1+\frac{n}{n_0}\right),$$

or

$$v = ci+\left(\frac{nv_0}{n_0}\right),$$

neglecting the small quantity (nci/n_0).

These three equations (two differential, one ordinary), determine the behaviour of the three unknowns v, i, x, in terms of the known speed change n.

The corresponding p-equations are

$$(mp^2+kp)x-bv = 0,$$

$$ai_0x-v+(Lp+R_0)i = 0,$$

$$v-ci = (nv_0/n_0).$$

As in Example 1, there is no need to know what terms occupy the right-hand side; but since v is the only quantity which can change suddenly with a sudden speed change n, the right-hand sides must actually be as shown.

Eliminating x and i, we find

$$\{(Lp+R_1)(mp^2+kp)+abci_0\}v = (Lp+R_0)(mp^2+kp)\frac{nv_0}{n_0},$$

where $R_1 = R_0-c$. Referring to Fig. 51 again, we observe that R_0 is the gradient of the line RR'; so R_1 is the difference between the gradients of RR' and CC'; it is positive.

The stability of the system therefore depends upon the properties of

$$(Lp+R_1)(mp^2+kp)+abci_0,$$

or $$(Lm)p^3+(R_1m+Lk)p^2+(R_1k)p+(abci_0).$$

Referring to Appendix V, we find that the conditions for this system to be stable are as follows (the p-expression has been written, as it must be, with the coefficient of p^3 positive):

$$(1)\ R_1m+Lk>0. \qquad (2)\ abci_0>0.$$

These two are automatically satisfied, for R_1, a, b and c are all positive.

$$(3)\ R_1k(R_1m+Lk)>Lmabci_0.$$

This last equation gives the 'safe' values of k. It may be re-written

$$R_1Lk^2+R_1^2mk-Lmabci_0>0,$$

and the only positive values of k which enable this to be satisfied are those for which

$$k>\frac{-R_1^2m+\sqrt{R_1^4m^2+4R_1L^2mabci_0}}{2R_1L}.$$

This is the required criterion of stability.

But absence of sustained oscillations does not in itself mean that a control system is satisfactory. Usually the engineer requires that the controlled quantity shall attain its final value in a critically damped manner; or perhaps he permits a little oscillation, because this allows the quantity to get near to its final value more quickly (Fig. 52). A convenient supposition, which gives about the right degree of damping, is that each oscillatory term $\cos \omega t$ or $\sin \omega t$ is damped by a factor $e^{-\omega t}$; the condition for this is also given in Appendix V. Applied to the present problem, this condition leads to the criterion:

$$(Lm)^2 d^2 - (R_1 m + Lk)^2 (R_1 k)^2 + 2(Lm)(R_1 k)^3 + 2d(R_1 m + Lk)^3 - 4(Lm)(R_1 m + Lk)(R_1 k) d = 0,$$

where $d = abci_0$. Rewriting the criterion as an equation for k, we obtain

$$-R_1^2 L^2 k^4 + 2L^3 dk^3 - (R_1^4 m^2 - 2R_1 L^2 md) k^2 + 2R_1^2 Lm^2 dk + (L^2 m^2 d^2 + 2R_1^3 m^3 d) = 0.$$

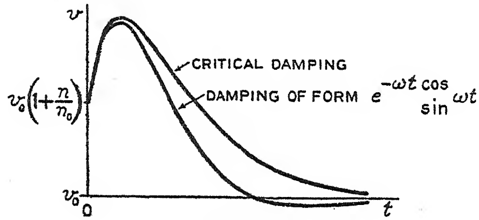

Fig. 52.

This equation of the fourth degree has a real positive root, which will be found to be somewhat larger than the value of k which ensures stability; large enough, in fact, to ensure the degree of damping which is required.

If the condition for critical damping is written down, it leads to the rather similar equation

$$-R_1^2 L^2 k^4 + (4L^3 d + 2R_1^3 Lm) k^3 - (R_1^4 m^2 + 6R_1 L^2 md) k^2 - 6R_1^2 Lm^2 dk + (27 L^2 m^2 d^2 + 4R_1^3 m^3 d) = 0,$$

and the positive real root of this equation will be larger still, for this is the stiffest of the three criteria which we have discussed.

CHAPTER VII

SOME PRACTICAL EXAMPLES

7·1. The reader of the preceding six chapters is now the possessor of a set of tools, new and shining, for prising open the mysteries of electrical circuits. But the circuits which we have discussed have been chosen so that each displayed chiefly the prowess of the newest tool; we have not yet had to lay the circuit on the bench while we scratched our heads dubiously before the tool-box, wondering which tool to select. In the examples of this chapter we shall pass through this final discipline. They are chosen from several branches of electrical engineering, and in each we shall have to consider what is the best of the methods available to us.

7·2. Ideal and Non-Ideal Interruption of Alternating Current.

Two taps on a transformer winding are connected by contactors to the extremities of a reactor, which is used in on-load tap-changing (Fig. 53). The reactor has inductance 100 millihenries, and equivalent shunt capacitance 1000 micromicrofarads; its

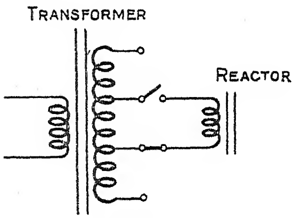

Fig. 53.

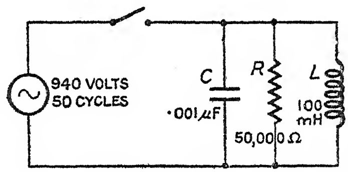

Fig. 54.

losses are equivalent to a parallel resistance of 50,000 ohms. The transformer reactance is negligible by comparison. Find the wave-form of voltage which appears across one of the contactors when it interrupts a current of 30 amps., 50 cycles,

(a) exactly at current zero;

(b) before current zero, when the instantaneous current value is 0·2 amp.

The relevant portions of the circuit are set out more clearly in Fig. 54.

The method of impedance operators is usually the quickest, unless the allowance for initial conditions is difficult. Here, the voltage which we want is zero until the instant of switch operation; so the value afterwards is the same as would be produced by injecting across the switch terminals into a dead circuit a current equal and opposite to that through the closed switch (Fig. 55).

Intelligent simplification should never be despised. We observe that the frequency of the (L, C) circuit is 15,900 cycles, whereas the supply frequency is 50 cycles. For many cycles of the (L, C) oscillation, therefore, the injected current will not diverge perceptibly from the straight line $I\Omega t$ (when interruption is at current zero), or $I\Omega t - I_0$ (for pre-zero interruption). This simplifies the working considerably, without any loss of accuracy so far as the interesting part of the transient period is concerned.

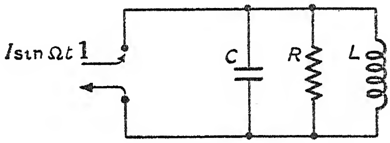

Fig. 55.

The operational impedance of the circuit is

$$\frac{1}{Cp+(1/R)+(1/Lp)}, \quad = \frac{RLp}{RLCp^2+Lp+R}.$$

If we work out problem (*b*), we can treat (*a*) as a special case. The voltage produced when $(I\Omega t - I_0) . \mathbf{1}$ is injected into this circuit is given by

$$\begin{aligned} v &= \frac{RLp}{RLCp^2+Lp+R} . (I\Omega t - I_0) . \mathbf{1} \\ &= \frac{RLp}{RLCp^2+Lp+R} . \left(\frac{I\Omega}{p} - I_0\right) . \mathbf{1} \\ &= \frac{1}{C} \frac{-I_0 p + I\Omega}{p^2+(1/RC)p+(1/LC)} . \mathbf{1}. \end{aligned}$$

Since the next step depends on whether the factors of the bottom line are real or complex, it is best now to substitute for R, L and C. Measuring time in microseconds, we obtain

$$\begin{aligned} v &= 1000 \frac{-I_0 p + I\Omega}{p^2+0{\cdot}02p+0{\cdot}01} . \mathbf{1} \\ &= 1000 \frac{-I_0 p + I\Omega}{(p+0{\cdot}01)^2+(0{\cdot}0995)^2} . \mathbf{1}; \end{aligned}$$

but in most work of this type we are not interested in the difference between 0·0995 and 0·1, so we write

$$v = 1000\frac{-I_0 p + I\Omega}{(p+0{\cdot}01)^2+(0{\cdot}1)^2}\cdot 1.$$

Turning now to the table of standard forms, we find that the interpretation is

$$v = 1000\left\{-\frac{I_0}{0{\cdot}1}e^{-0{\cdot}01t}\sin(0{\cdot}1t) + \frac{I\Omega}{0{\cdot}01}\left[1-e^{-0{\cdot}01t}\left(\cos[0{\cdot}1t]+\frac{0{\cdot}01}{0{\cdot}1}\sin[0{\cdot}1t]\right)\right]\right\}.$$

In problem (*a*), $I_0 = 0$, $I = 30\sqrt{2}$, $\Omega = 3{\cdot}14\times10^{-4}$ (when time is measured in microseconds). Therefore

$$v = 940\sqrt{2}\{1-e^{-0{\cdot}01t}[\cos(0{\cdot}1t)+\tfrac{1}{10}\sin(0{\cdot}1t)]\}.$$

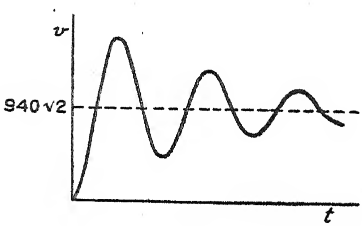

Fig. 56.

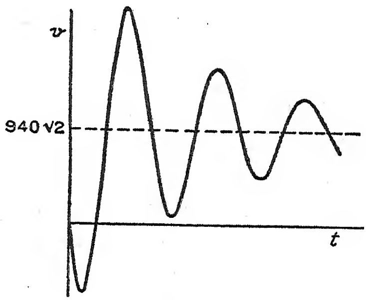

Fig. 57.

The form of this voltage is shown in Fig. 56. The oscillation is dying down to the steady value $940\sqrt{2}$, which is the crest 50 cycle voltage across the open contacts. This is natural, since from the power frequency viewpoint the circuit is practically pure inductance.

In problem (*b*) there is superimposed upon this the contribution of the I_0 term, which is

$$-2000e^{-0{\cdot}01t}\sin(0{\cdot}1t).$$

The resultant shape is shown in Fig. 57. It makes a lot of difference whether the current is extinguished at 0·2 amp. or 0·0 amp.

7·3. Recovery Voltage upon Extinction of an Earth Fault in a Supply System Earthed through a Petersen Coil.

A three-phase star-connected transformer supplies a transmission system, each of whose lines has capacitance K to the other lines, and $C/3$ to earth. The transformer neutral is earthed through an inductance L (known as a Petersen coil), whose value is such that $1/2\pi\sqrt{LC}$ is nearly equal to the supply frequency. The losses in the coil and system can be represented by a resistance R in parallel with the coil (Fig. 58). Compared with L, the transformer reactance is negligible.

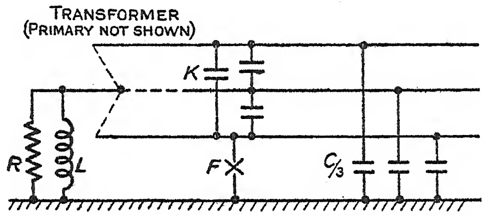

Fig. 58.

At the point F an earth fault has become established, for a sufficient time for the current in the fault to have attained a steady (alternating) value. This current becomes extinguished at a zero point; we desire to know the wave-form of voltage appearing across the fault.

This problem is made to look much simpler if we replace it by the equivalent one in which a current equal and opposite to the fault current is injected into the circuit at the fault *with all constant voltage generators regarded as having zero impedance*. For the transformer here plays the part of a constant voltage generator; if it is assigned zero impedance, the capacitances K are short-circuited and disappear from the problem, while the capacitances $C/3$ are connected in parallel. Thus the problem can be represented in exactly the form shown in Fig. 55.

This makes the problem look very like the last one, with the significant difference that the natural frequency of the circuit is now nearly the same as the supply frequency. We could start by working out the value of the steady fault current; but we save ourselves this labour by calling it $I \sin \Omega t$, and resolving to give I such a value that the ultimate voltage equals the transformer phase voltage—say $V \cos(\Omega t+\theta)$.

The operational impedance of the circuit is

$$\frac{RLp}{RLCp^2+Lp+R}.$$

Therefore the voltage produced by injecting $I\sin\Omega t\,.\,\mathbf{1}$ is given by

$$v = \frac{RLp}{RLCp^2+Lp+R}.\{I\sin\Omega t\,.\,\mathbf{1}\}$$

$$= \frac{I\Omega}{C}\frac{p^2}{[(p+\alpha)^2+\omega^2][p^2+\Omega^2]}.\mathbf{1},$$

where $\qquad 2\alpha = 1/RC, \quad \alpha^2+\omega^2 = 1/LC.$

Splitting up the p-expression into partial fractions, we suppose

$$\frac{p^2}{[(p+\alpha)^2+\omega^2][p^2+\Omega^2]} \equiv \frac{Ap^2+Bp}{(p+\alpha)^2+\omega^2}+\frac{Ep^2+Fp}{p^2+\Omega^2}+G;$$

then, multiplying both sides by $[(p+\alpha)^2+\omega^2][p^2+\Omega^2]$, we obtain

$A = -(\alpha^2+\omega^2-\Omega^2)/D$, where $D = (\alpha^2+\omega^2-\Omega^2)^2+4\alpha^2\Omega^2$;

$B = -2\alpha(\alpha^2+\omega^2)/D$;

$E = (\alpha^2+\omega^2-\Omega^2)/D$;

$F = 2\alpha\Omega^2/D$;

$G = 0$.

Substituting these values, and performing the p-operations with the aid (if necessary) of Appendix I, we find

$$v = \frac{I\Omega}{CD}\Big\{(\alpha^2+\omega^2-\Omega^2)(\cos\Omega t-e^{-\alpha t}\cos\omega t)$$
$$+2\alpha\Omega\sin\Omega t-\frac{\alpha}{\omega}(\alpha^2+\omega^2+\Omega^2)e^{-\alpha t}\sin\omega t\Big\}.$$

We have still to find the value of I in terms of V. When t becomes very large, v tends towards the value

$$\frac{I\Omega}{CD}\{(\alpha^2+\omega^2-\Omega^2)\cos\Omega t+2a\Omega\sin\Omega t\}, \qquad = \frac{I\Omega}{C\sqrt{D}}\cos(\Omega t+\theta),$$

where $\quad \cos\theta = (\alpha^2+\omega^2-\Omega^2)/\sqrt{D}, \quad \sin\theta = -2\alpha\Omega/\sqrt{D}$;

the bottom line has of course been adjusted to make

$$\cos^2\theta+\sin^2\theta = 1.$$

Therefore $I\Omega/C\sqrt{D}$ must equal V; and we may rewrite v in the form

$$v = V\left\{\frac{(\alpha^2+\omega^2-\Omega^2)(\cos\Omega t - e^{-\alpha t}\cos\omega t) + 2\alpha\Omega\sin\Omega t - \dfrac{\alpha}{\omega}(\alpha^2+\omega^2+\Omega^2)\,e^{-\alpha t}\sin\omega t}{\sqrt{(\alpha^2+\omega^2-\Omega^2)^2+4\alpha^2\Omega^2}}\right\}.$$

A more seemly form is

$$v = V\{\cos(\Omega t+\theta) - Ke^{-\alpha t}\cos(\omega t+\phi)\},$$

where, it will be found,

$$K = \sqrt{(\alpha^2+\omega^2)}/\omega,$$
$$\cos\phi = \omega\,(\alpha^2+\omega^2-\Omega^2)/\sqrt{D(\alpha^2+\omega^2)},$$
$$\sin\phi = -\alpha(\alpha^2+\omega^2+\Omega^2)/\sqrt{D(\alpha^2+\omega^2)}.$$

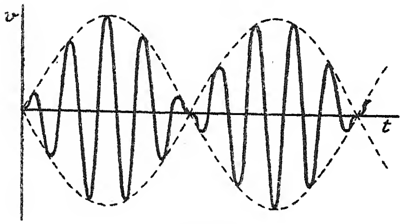

Fig. 59.

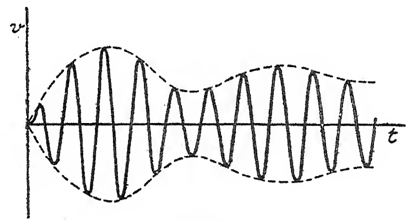

Fig. 60.

To gain an idea of the form of this voltage, we must bear in mind that α/ω is quite small—not greater, probably, than 0·04. For a preliminary survey, therefore, we put $\alpha = 0$, and obtain

$$\begin{aligned} v &= V(\cos\Omega t - \cos\omega t) \\ &= -2V\sin\left(\frac{\Omega+\omega}{2}t\right)\sin\left(\frac{\Omega-\omega}{2}t\right). \end{aligned}$$

Since ω is nearly equal to Ω, $\left(\dfrac{\Omega-\omega}{2}\right)$ represents quite a low frequency compared with $\left(\dfrac{\Omega+\omega}{2}\right)$; and the appearance of the wave is therefore that of a wave of frequency $\dfrac{1}{2\pi}\times\left(\dfrac{\Omega+\omega}{2}\right)$, whose amplitude varies at the frequency $\dfrac{1}{2\pi}\times\left(\dfrac{\Omega-\omega}{2}\right)$. The maximum value is nearly $2V$ (Fig. 59). The actual wave will start as in Fig. 59, but the pulsations will gradually fade out into a steady 50 cycle voltage (Fig. 60).

7·4. Modification of a Lightning Surge Voltage by the Cable System of a Power Station.

The busbars B of a power station supply three feeders, BA, BC, BD (Fig. 61). BA, BC consist of cables, respectively 150 and 300 yards long, connected at A and C to overhead lines; BD is a cable 10 miles in length.

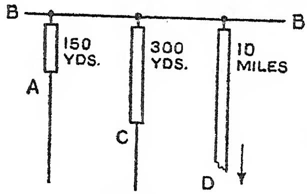

Fig. 61.

A lightning stroke to the overhead line near the point A causes a high voltage of approximately 'unit function' shape to be applied to the system at A. It is required to find the voltage on the busbars B, under the following assumptions:

(1) That the two short cables behave approximately as though half their capacitance were concentrated at each end, with their inductance between; actually, of course, these are distributed, a little bit to each bit of cable.

(2) That the cable AB has a total series inductance of 30 microhenries, and a total capacitance to earth of $\frac{1}{30}$ microfarad; and that the other cables have the same inductance L and capacitance C per unit length as AB.

(3) That the cable BD, which acts more or less as a continuous absorber of energy, may be treated as equivalent to a resistance of ohmic value $\sqrt{(L/C)}$. (With the values of L and C already given, this comes to 30 ohms.)

(4) That the overhead line at C exerts a negligible effect on the behaviour of the cable, i.e. C behaves like an open circuit.

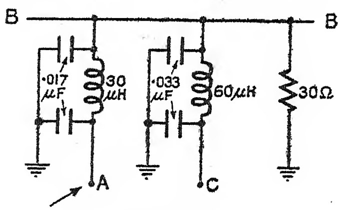

Fig. 62.

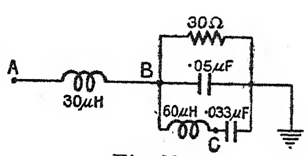

Fig. 63.

Under these assumptions, we shall be replacing the cable system by the network shown in Fig. 62; this is redrawn in a clearer form in Fig. 63, where the capacitance between the

point A and earth has been omitted, since it is in parallel with the whole system, and makes no difference to the voltage distribution.

The operational impedance Z between the point B and earth is given by

$$\frac{1}{Z} = \frac{1}{60p + (1/0{\cdot}033p)} + 0{\cdot}05p + \frac{1}{30},$$

if we work in microhenries and microfarads; which reduces to

$$\frac{3p^3 + 2p^2 + 2{\cdot}5p + 1}{30(2p^2 + 1)}.$$

Therefore the voltage v at the point B, worked out just as though this were an ordinary A.C. circuit, is given by

$$\begin{aligned} v &= V\frac{Z}{30p + Z}.1 \\ &= V\frac{2p^2 + 1}{3p^4 + 2p^3 + 4{\cdot}5p^2 + p + 1}.1. \end{aligned}$$

To interpret this expression, we need to find the factors of the bottom line. The method is given in Appendix V.

We write

$$3p^4 + 2p^3 + 4{\cdot}5p^2 + p + 1$$

in the form

$$3\{(p^2 + \tfrac{1}{3}p + f)^2 - (gp + h)^2\},$$

and proceed to discover suitable values of f, g and h. Equating corresponding terms on the two sides, we find

$$2f - g^2 = \tfrac{25}{18},$$

$$2f - 6gh = 1,$$

$$f^2 - h^2 = \tfrac{1}{3}.$$

The first and last of these equations give g^2 and h^2 in terms of f; the second shows that $36g^2h^2 = (2f-1)^2$. We can therefore eliminate g and h and obtain an equation for f, which turns out to be

$$216f^3 - 162f^2 - 60f + 47 = 0.$$

The first equation also shows that g will only be real if f is greater than $\frac{25}{36}$; of the three roots of the cubic equation for f, only one satisfies this condition; the equation is solved as an example in Appendix V, and the root is there found to be 0·709. We can then find at once that $g = 0{\cdot}169$, $h = 0{\cdot}411$, so

$$\begin{aligned}3p^4+2p^3+4{\cdot}5p^2+p+1 &\\ &\equiv 3\{(p^2+\tfrac{1}{3}p+0{\cdot}709)^2-(0{\cdot}169p+0{\cdot}411)^2\}\\ &= 3(p^2+0{\cdot}502p+1{\cdot}120)(p^2+0{\cdot}164p+0{\cdot}298).\end{aligned}$$

These two quadratic factors cannot be split into real factors of the first degree.

We now seek partial fractions, in the following form:

$$\frac{2p^2+1}{3(p^2+0{\cdot}502p+1{\cdot}120)(p^2+0{\cdot}164p+0{\cdot}298)} \equiv \frac{Ap^2+Bp}{p^2+0{\cdot}502p+1{\cdot}120}+\frac{Cp^2+Dp}{p^2+0{\cdot}164p+0{\cdot}298}+E.$$

Multiplying both sides by

$$3(p^2+0{\cdot}502p+1{\cdot}120)(p^2+0{\cdot}164p+0{\cdot}298),$$

and equating corresponding terms, we find

$$A=-0{\cdot}533,\quad B=-0{\cdot}064,\quad C=-0{\cdot}467,\quad D=-0{\cdot}280,\quad E=1.$$

(The last named may be more simply obtained by putting $p = 0$.)

Therefore

$$\begin{aligned}v &= V\left\{1-\frac{0{\cdot}533p^2+0{\cdot}064p}{(p+0{\cdot}251)^2+(1{\cdot}028)^2}-\frac{0{\cdot}467p^2+0{\cdot}280p}{(p+0{\cdot}082)^2+(0{\cdot}540)^2}\right\}.1\\ &= V\{1-e^{-0{\cdot}251t}(0{\cdot}533\cos 1{\cdot}028t-0{\cdot}068\sin 1{\cdot}028t)\\ &\qquad -e^{-0{\cdot}082t}(0{\cdot}467\cos 0{\cdot}540t+0{\cdot}448\sin 0{\cdot}540t)\},\end{aligned}$$

which is perhaps more conveniently written

$$\begin{aligned}v = V\{1 &-0{\cdot}537\,e^{-0{\cdot}251t}\cos(1{\cdot}028t+0{\cdot}127)\\ &-0{\cdot}647\,e^{-0{\cdot}082t}\cos(0{\cdot}540t-0{\cdot}764)\}.\end{aligned}$$

The form of this voltage is shown in Fig. 64 (full curve). Its maximum value is 1·35 times the applied voltage.

Now suppose that the applied voltage, instead of being of 'unit function' shape, rises steeply to its maximum and then falls to half that value in 20 microseconds; what happens to the

voltage at B? This solution may be deduced from the other with the help of Duhamel's integral. The applied voltage, instead of being $V.\mathbf{1}$, will now be $Ve^{-0\cdot035t}\times\mathbf{1}$.

If we apply the Duhamel method in the obvious way, we shall reach a solution something like the solution already found. Plotting such solutions is tedious, as the reader will discover if he tries it. Since we have already plotted this solution in Fig. 64, why not deduce the other by numerical integration—an easy arithmetical job?

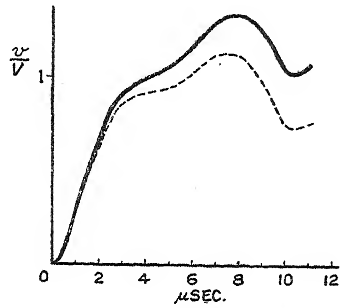

Fig. 64.

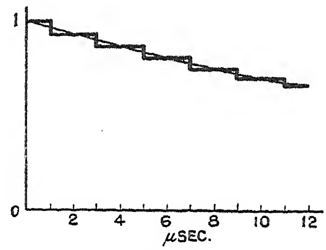

Fig. 65.

The essential of the Duhamel method is to represent the shock on the circuit by a series of 'unit functions'. In this instance, it will suffice if we apply the successive 'unit functions' at intervals of 2 microseconds; but, in order to make the built-up curve zigzag on both sides of the true one (see Fig. 65), we apply the step, which lowers the curve from its 0-microsecond to its 2-microsecond value, at time 1 microsecond. Subsequent steps then occur at 3, 5, 7, ... microseconds. Omitting the constant factor V, the staircase which we use to represent $e^{-0\cdot035t}\times\mathbf{1}$ is therefore as follows:

At time	0	microseconds,		$+1.$
,,	1	,,	$-(1-e^{-0\cdot07})$	$=-0\cdot067_6.$
,,	3	,,	$-(e^{-0\cdot07}-e^{-0\cdot14})$	$=-0\cdot063_0.$
,,	5	,,	$-(e^{-0\cdot14}-e^{-0\cdot21})$	$=-0\cdot058_8.$
,,	7	,,	$-(e^{-0\cdot21}-e^{-0\cdot28})$	$=-0\cdot054_8.$
,,	9	,,	$-(e^{-0\cdot28}-e^{-0\cdot35})$	$=-0\cdot051_1.$

The response of the system to this staircase is as follows:

Component	(Time μsec.)					
	1	2	3	4	5	6
1	0·252	0·666	0·921	1·007	1·071	1·191
$-0{\cdot}067_6$	0	−0·017	−0·045	−0·062	−0·068	−0·072
$-0{\cdot}063_0$	—	—	0	−0·016	−0·042	−0·058
$-0{\cdot}058_8$	—	—	—	—	0	−0·015
$-0{\cdot}054_8$	—	—	—	—	—	—
$-0{\cdot}051_1$	—	—	—	—	—	—
Total	0·252	0·649	0·876	0·929	0·961	1·046
Component	7	8	9	10	11	
1	1·315	1·342	1·236	1·048	1·100	
$-0{\cdot}067_6$	−0·081	−0·089	−0·091	−0·084	−0·071	
$-0{\cdot}063_0$	−0·063	−0·067	−0·075	−0·083	−0·085	
$-0{\cdot}058_8$	−0·039	−0·054	−0·059	−0·063	−0·070	
$-0{\cdot}054_8$	0	−0·014	−0·036	−0·050	−0·055	
$-0{\cdot}051_1$	—	—	0	−0·013	−0·034	
Total	1·132	1·118	0·975	0·755	0·785	

The first component is read off from the full curve of Fig. 64. The others are simply obtained by multiplying the figures in the top line by the appropriate factor, and also shifting them to the right by the required time-interval. The whole process only requires a few minutes of slide-rule work, once the response to unit function has been calculated; the total, giving the response to $Ve^{-0{\cdot}035t}\times \mathbf{1}$, is plotted as a dotted curve in Fig. 64.

We have been using Duhamel's integral in the form

$$u(t) = U(0)\,u_1(t)+\int_0^t U'(\tau)\,u_1(t-\tau)\,d\tau,$$

or

$$U(0)\,u_1(t)+\sum_0^t U'(\tau)\,\delta\tau\,.\,u_1(t-\tau).$$

$U(0)\,u_1(t)$ is the first component; $U'(\tau)\,\delta\tau$ is the height of the 'step'—which, however, we have located at the instant $(\tau-1)$, so that instead of $u_1(t-\tau)$ we have used $u_1(t-\tau+1)$.

7·5. Behaviour of a Six-Phase Rectifier Circuit.

A six-phase rectifier equipment of negligible internal impedance supplies the terminals A, B of a smoothing circuit (L, C), which is loaded with a resistive load R (Fig. 66). The voltage applied to A, B is the envelope of a series of 50-cycle sine-waves, of crest

value V, displaced by 60° (Fig. 67); but the rectifier can only pass current in the direction shown by the arrow, so that if the current tends to reverse it becomes extinguished. The terminals are then effectively disconnected from the voltage supply until the voltage across them falls lower than the instantaneous value of the supply voltage, when the arc re-strikes.

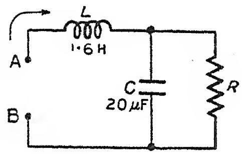

Fig. 66.

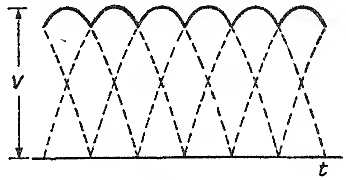

Fig. 67.

For very light loads, the voltage across C and R will remain almost constant at the value V. For heavy loads, the arc will remain in being throughout the cycle, and the mean voltage v_m across C and R will equal the mean value of the rectified voltage, which is $3V/\pi$. We shall therefore expect v_m to be related to the mean load current i_m by some such curve as is shown in Fig. 68. It is required to find, for the values of L and C shown in Fig. 66, the point P at which the voltage starts to rise; that is, to find the load at which the rectifier current just touches zero.

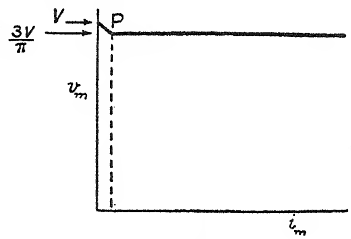

Fig. 68.

It is perfectly possible to obtain an exact formula for the wave-form of rectifier current, by using the method set forth in § 4·7, Example 2. But unless the reader has a taste for several hours of algebra, he had better start with a bit of intelligent simplification, as follows.

The voltage wave shown in Fig. 67 oscillates between the values V and $0{\cdot}866V$, at a frequency of 300 cycles. For a rough approximation, replace the wave by a sinusoidal ripple of the same maximum and minimum, and the same frequency (Fig. 69). The formula for this wave will be

$$V(0{\cdot}933 + 0{\cdot}067 \sin 6\Omega t),$$

where
$$\Omega = 2\pi \times 50$$
$$= 314, \text{ if time is measured in seconds.}$$

At 300 cycles, L has impedance 3014 ohms; C has 26 ohms. Therefore, whatever the value of R, the 300-cycle impedance of the circuit must lie between 3014 and 2988 ohms. Calling it 3000, we see that the peak alternating current through L is

$$(0{\cdot}067V/3000) \text{ amps.}$$

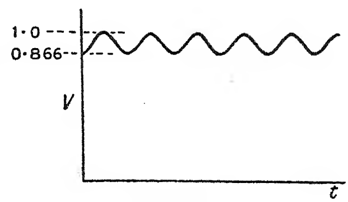

Fig. 69.

The direct current depends only on R, and is

$$(0{\cdot}933V/R) \text{ amps.}$$

If the direct current is just large enough to prevent the current from tending to reverse, these must be equal; from which we find

$$R = 42{,}000 \text{ ohms (approximately).}$$

The critical damping resistance of the (L, C) circuit is

$$\frac{1}{2}\sqrt{\frac{L}{C}} = 141 \text{ ohms.}$$

Rough though this method is, it makes it quite clear that the interesting value of R is far larger than the critical resistance; so much larger, that it exerts only the slightest influence on the oscillations of the (L, C) circuit, and is really effective only as a drain for direct current. This suggests the following method of attack:

(1) Find the current through L when the wave-form of Fig. 67 is applied to the circuit of Fig. 66, with R open-circuited.

(2) Find the greatest negative value of this current, and hence ascertain what value must be assigned to R so that the direct current just cancels this negative A.C. peak current. (For this purpose, the voltage across R may be assumed to have the mean value of the voltage form in Fig. 67; namely, $3V/\pi$.)

In calculating the current when R is open-circuited, it would not be wise to use the method of § 4·7, Example 2; for, in the absence of damping, the effect of each voltage cycle lasts for ever, so that the cycle which happened an hour ago may have a residual effect as large as the cycle which has just been completed. We shall adopt a method which is less elegant, but more safe.

Take as variables the current i in L, and the voltage v on C (Fig. 70). The applied voltage may be taken as $V\sin(\Omega t+\pi/3)$, during the interval from $t=0$ to $t=\pi/3\Omega$. At the beginning of the interval, let $i=i_0$, and $v=v_0$.

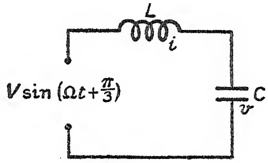

Fig. 70.

Writing down the p-equations by the method of § 5·5 (b), we obtain

$$Lp(i-i_0)+v=V\sin(\Omega t+\pi/3),$$

$$Cp(v-v_0)=i.$$

Eliminating v between these,

$$(LCp^2+1)\,i=LCp^2i_0-Cpv_0+CpV\sin(\Omega t+\pi/3),$$

or
$$(p^2+\omega^2)\,i=p^2i_0-\frac{p}{L}v_0+\frac{p}{L}V\sin(\Omega t+\pi/3),$$

where
$$\omega^2=1/LC.$$

Thus, writing $V\sin(\Omega t+\pi/3)$ in its operational form, we obtain the operational form for i; namely,

$$i=\left\{\frac{i_0p^2}{p^2+\omega^2}-\frac{v_0}{L}\frac{p}{p^2+\omega^2}+\frac{V}{2L}\frac{\sqrt{3}\,p^3+\Omega p^2}{(p^2+\omega^2)(p^2+\Omega^2)}\right\}.1.$$

Writing the last term in partial fractions as usual, and interpreting, we find

$$i=i_0\cos\omega t-\frac{v_0}{L\omega}\sin\omega t$$
$$+\frac{V}{2L(\Omega^2-\omega^2)}\{\Omega(\cos\omega t-\cos\Omega t)-\omega\sqrt{3}\sin\omega t+\Omega\sqrt{3}\sin\Omega t\}.$$

We now have to find i_0 and v_0 by writing down the conditions that i and v may have the same values when $t=0$ and $t=t_1$ (where $t_1=\pi/3\Omega$); for the circuit must be assumed to have attained a steady state. These conditions are

$$i=i_0 \text{ when } t=t_1,$$

and
$$\int_0^{t_1} i\,dt=0;$$

for the integral of i over any interval of time is equal to the charge acquired in that time by the condenser. In this way, we find

$$i_0(1-\cos\omega t_1)+\frac{v_0}{L\omega}\sin\omega t_1$$
$$=\frac{V}{2L(\Omega^2-\omega^2)}\{\Omega+\Omega\cos\omega t_1-\omega\sqrt{3}\sin\omega t_1\},$$

and
$$i_0\sin\omega t_1-\frac{v_0}{L\omega}(1-\cos\omega t_1)$$
$$=\frac{V}{2L(\Omega^2-\omega^2)}\{\omega\sqrt{3}(1-\cos\omega t_1)-\Omega\sin\omega t_1\}.$$

Solving these equations, we obtain

$$i_0 = 0,$$
$$v_0=\frac{\omega V}{2(\Omega^2-\omega^2)}\frac{\Omega\sin\omega t_1-\omega\sqrt{3}(1-\cos\omega t_1)}{(1-\cos\omega t_1)}$$
$$=\frac{\omega V}{2(\Omega^2-\omega^2)}\frac{\Omega\cos(\omega t_1/2)-\omega\sqrt{3}\sin(\omega t_1/2)}{\sin(\omega t_1/2)}.$$

Hence, after a little trigonometry, we find

$$i=\frac{\Omega V}{2L(\Omega^2-\omega^2)\sin(\omega t_1/2)}\left\{\sin\left(\frac{\omega t_1}{2}-\omega t\right)-2\sin\frac{\omega t_1}{2}\cos\left(\Omega t+\frac{\pi}{3}\right)\right\}.$$

A simpler form may be obtained if we measure time from the instant when the applied wave reaches its crest. Using electrical angles instead of time-units, let

$$\Omega t-\pi/6=\theta,$$

and
$$\omega/\Omega=\lambda.$$

The expression now becomes

$$i=\frac{V}{L\Omega(1-\lambda^2)}\left\{\sin\theta-\frac{\sin\lambda\theta}{2\sin(\lambda\pi/6)}\right\},$$

where θ varies from $-\pi/6$ to $+\pi/6$.

With the particular component values given in Fig. 66, $\omega = 177$, so $\lambda = 0{\cdot}564$. Therefore

$$i = \frac{1{\cdot}466V}{L\Omega}\{\sin\theta - 1{\cdot}7183\sin(0{\cdot}564\theta)\},$$

and the form of this current is plotted in Fig. 71. The negative crest occurs when θ is about $-17°$, and its value is $-0{\cdot}00927V/L\Omega$.

The mean direct current through L is $3V/\pi R$. If this is just enough to nullify the negative A.C. crest, we must have

$$R = 51{,}800 \text{ ohms}.$$

It would of course be possible to allow for the fact that the voltage across R is really v, the voltage occurring in our equations; but it is doubtful whether such refinements can be justified in this approximate method. Anyway, the instant when i attains its crest must be quite close to the zero of the oscillation on v; so the instantaneous voltage across R will be very near to the mean value of v, which must be $3V/\pi$.

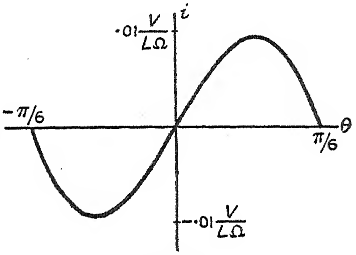

Fig. 71.

7·6. Surge Propagation in Alternator Windings.

One phase of an alternator winding consists of 24 coils connected in series, each with self-inductance L and each having a capacitance C to the earthed core. The capacitance is really distributed

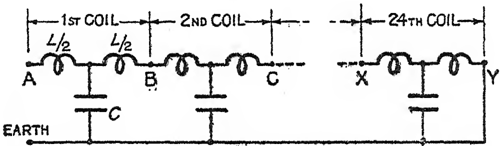

Fig. 72.

along the conductors of the coil, but with sufficient accuracy it may be considered to be concentrated at the centre of each coil, with half the inductance on either side of it (Fig. 72). If a surge voltage of 'unit function' form is applied between terminal A and earth, what is the form of the voltage to earth at B, the junction of the first and second coils?

It is to be hoped that the reader will not attempt to write down the complete p-equation for this circuit, since it is of the 48th degree. It is useless to attempt a problem like this unless some simpler method can be found; for instance, the following.

The winding between the points A and 'Earth' will have an operational impedance, Z. The last 23 coils, to the right of the point B, will have practically the same impedance; for what is one coil among so many? The condition for the two impedances to be equal is

$$\frac{Lp}{2}+\cfrac{1}{Cp+\cfrac{1}{\cfrac{Lp}{2}+Z}}=Z;$$

and this is an equation for Z, whose solution is

$$Z=\sqrt{\frac{L}{C}\left(1+\frac{LC}{4}p^2\right)}.$$

(The reader will see that what we have really done is to assume that the number of coils is infinite.)

The circuit has now been simplified to that shown in Fig. 73. The current in AM is $\frac{V}{Z}.1$; in MB it is $\frac{v}{Z}$; and the difference of these is the current taken by the condenser C,

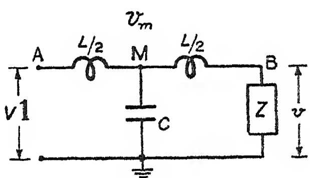

Fig. 73.

$$=Cpv_m$$

$$=Cp\left(V.1-\frac{Lp}{2Z}V.1\right).$$

Therefore $$\frac{V}{Z}.1-\frac{v}{Z}=Cp\left(1-\frac{Lp}{2Z}\right)V.1,$$

or $$v=(1-ZCp+\tfrac{1}{2}LCp^2)\,V.1.$$

This bears no resemblance to any 'standard form' which we have discussed so far. Nevertheless we can get quite a long way by simply applying our known rules of interpretation—that is, by expanding in powers of $1/p$ and obtaining the solution in the form of an infinite series.

Substituting for Z, we find

$$v = V\{1+\tfrac{1}{2}LCp^2 - \sqrt{LCp^2(1+\tfrac{1}{4}LCp^2)}\}.1.$$

Writing $1/\omega^2$ for LC, as we have usually done,

$$v = V\left\{1+\frac{p^2}{2\omega^2} - \sqrt{\frac{p^2}{\omega^2}\left(1+\frac{p^2}{4\omega^2}\right)}\right\}.1.$$

By taking out the factor $p^2/2\omega^2$, we may transform the bracket to a form which contains $1/p$ instead of p; thus,

$$v = \frac{Vp^2}{2\omega^2}\left\{1+\frac{2\omega^2}{p^2} - \sqrt{1+\frac{4\omega^2}{p^2}}\right\}.1.$$

Now use the Binomial Theorem to expand the square root in powers of $(4\omega^2/p^2)$—as explained in Appendix II. We then obtain

$$v = \frac{Vp^2}{2\omega^2}\left\{1+\frac{2\omega^2}{p^2} - \left[1+\frac{1}{2}\frac{4\omega^2}{p^2} - \frac{1.1}{2^2.2!}\left(\frac{4\omega^2}{p^2}\right)^2 + \frac{1.1.3}{2^3.3!}\left(\frac{4\omega^2}{p^2}\right)^3 - \ldots\right]\right\}.1,$$

$$= \frac{Vp^2}{2\omega^2}\left\{\frac{1}{2!}\left(\frac{2\omega^2}{p^2}\right)^2 - \frac{1.3}{3!}\left(\frac{2\omega^2}{p^2}\right)^3 + \frac{1.3.5}{4!}\left(\frac{2\omega^2}{p^2}\right)^4 - \ldots\right\}.1.$$

The general term of this series, including the factor $p^2/2\omega^2$, is

$$(-1)^n \frac{1.3.5\ldots(2n-3)}{n!}\left(\frac{2\omega^2}{p^2}\right)^{n-1}.$$

If we multiply the top and bottom lines by $2.4.6\ldots.(2n-2)$, which is $2^{n-1}\times(n-1)!$, we obtain the simpler form

$$(-1)^n \frac{(2n-2)!}{(n-1)!\,n!}\left(\frac{\omega^2}{p^2}\right)^{n-1}.$$

Therefore $$v = V\left\{\frac{2!}{1!\,2!}\frac{\omega^2}{p^2} - \frac{4!}{2!\,3!}\frac{\omega^4}{p^4} + \frac{6!}{3!\,4!}\frac{\omega^6}{p^6} - \ldots\right\}.1$$

$$= V\left\{\frac{(\omega t)^2}{1!\,2!} - \frac{(\omega t)^4}{2!\,3!} + \frac{(\omega t)^6}{3!\,4!} - \frac{(\omega t)^8}{4!\,5!} + \ldots\right\}.$$

Here we have obtained v in the form of a series of quite a simple type; the amount which we can discover about the behaviour of v is now only limited by our patience in working out the terms of

the series. Fig. 74 shows the behaviour of v up to the time given by $\omega t = 5$. It shows that the voltage at B will reach a crest value which is about 13 per cent in excess of the applied voltage.

This example has been included in order to show that, even if we run into operational expressions of unfamiliar kinds, they will probably yield to the familiar rules. In some cases, the expert is able to detect that the series which looks so fearsome corresponds with some well-known and oft-tabulated function; thus, in the present example, we may write

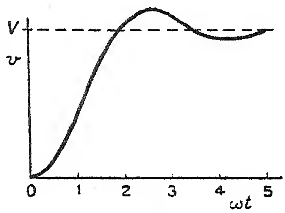

Fig. 74.

$$v = V\left(1 - \frac{J_1(2\omega t)}{\omega t}\right),$$

or else $$v = 2V\int_0^t \frac{J_2(2\omega t)}{t}\,dt,$$

where $J_1(2\omega t)$, $J_2(2\omega t)$ are functions called 'Bessel Functions of orders 1 and 2', whose numerical values may be found in mathematical tables. The functions which correspond with a few of these more recondite p-expressions have been included, for completeness, in Appendix I; the reader may obtain the second of the above expressions for v, by the use of forms (1) and (16).

7·7. With this example, the reader has been conducted to the very edge of our allotted territory. Like Alice on the mantelpiece, we stand with one foot in Looking-glass Land, where p behaves in strange ways and needs treating strangely. Alice got into trouble with Looking-glass arithmetic—'"She ca'n't do sums a *bit*!" the Queens said together, with great emphasis'—and we might be no better if we ventured further. This book is not a guide to Looking-glass Land; in plainer words, it is not a guide to the behaviour of all circuits, but only to those simple, useful circuits which are mentioned in § 1·1. But I hope that the reader has found that even this restricted territory is varied enough to be interesting, for there is no doubt at all about its usefulness.

APPENDIX I

OPERATORS—STANDARD FORMS

The inclusion or exclusion of forms in this list must needs be somewhat arbitrary. The majority of useful forms are interpretations of expressions like $F(p).1$ (or $F(p).\mathbf{1}$); and for the backbone of the list, $F(p)$ covers a more or less complete series of the partial-fraction forms given in Chapter III. A few derived forms clamour to be introduced, because of the simplicity of their t-expressions; these have their reference numbers inset, and distinguished by the letter (a). To interpret his partial fractions, the reader should look through the main sequence of numbered forms, until he finds those which he requires; but he should glance at the neighbouring (a) forms, in case his problem is one which terminates in this happy simplicity.

The following have been *excluded*:

(1) Derived forms which do not lead to simple t-expressions; thus, from

$$\frac{p^2}{p^2+\omega^2}.1 = \cos \omega t$$

we may deduce

$$\frac{1}{p^2+\omega^2}.1 = \frac{1-\cos \omega t}{\omega^2},$$

but we do not list this, since $1/p^2+\omega^2$ is not a recommended partial fraction.

(2) Forms designed to enable lazy people to avoid making partial fractions, like

$$\frac{p}{(p+\alpha)(p+\beta)}.1.$$

If the reader wants an extended list with these forms added, he can easily compile one for himself.

The operational equivalents of certain Bessel functions, useful in examples like the one in § 7·6, have been added. There are also a few operations of the form

$$\frac{1}{p+\alpha}f(t),$$

to facilitate the use (where convenient) of the method given in § 2·7. Many forms have been left out altogether, because they lie right outside the scope and methods of this book.

Basic Forms

(1) $$\frac{1}{p}\cdot f(t) = \int_0^t f(t)\,dt.$$

(2) $$p\cdot f(t) = f'(t), \quad \text{if } f(0) = 0.$$

Operations upon 1 (or 1)

(3) $$\frac{1}{p}\cdot 1 = t.$$

(4) $$\frac{1}{p^n}\cdot 1 = \frac{t^n}{n\,!}.$$

(5) $$\frac{p}{p+\alpha}\cdot 1 = e^{-\alpha t}.$$

(6) $$\frac{p}{(p+\alpha)^n}\cdot 1 = \frac{t^{n-1}}{(n-1)\,!}e^{-\alpha t}.$$

(7) $$\frac{p^2}{p^2+\omega^2}\cdot 1 = \cos\omega t.$$

(8) $$\frac{p}{p^2+\omega^2}\cdot 1 = \frac{1}{\omega}\sin\omega t.$$

(9) $$\frac{p^2}{(p^2+\omega^2)^2}\cdot 1 = \frac{t}{2\omega}\sin\omega t.$$

(10) $$\frac{p}{(p^2+\omega^2)^2}\cdot 1 = \frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t).$$

(10a) $$\frac{p^3-\omega^2 p}{(p^2+\omega^2)^2}\cdot 1 = t\cos\omega t.$$

(11) $$\frac{p^2}{(p+\alpha)^2+\omega^2}\cdot 1 = e^{-\alpha t}\left(\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\right).$$

(12) $$\frac{p}{(p+\alpha)^2+\omega^2}\cdot 1 = \frac{1}{\omega}e^{-\alpha t}\sin\omega t.$$

(12a) $$\frac{p^2+\alpha p}{(p+\alpha)^2+\omega^2}\cdot 1 = e^{-\alpha t}\cos\omega t.$$

(13) $$\frac{p}{\sqrt{(p^2+\alpha^2)}}\left(\frac{\sqrt{(p^2+\alpha^2)}-p}{\alpha}\right)^n.1 = J_n(\alpha t).$$

(13a) $$\frac{p}{\sqrt{(p^2+\alpha^2)}}.1 = J_0(\alpha t).$$

(14) $$\frac{p}{\sqrt{(p^2-\alpha^2)}}\left(\frac{p-\sqrt{(p^2-\alpha^2)}}{\alpha}\right)^n.1 = I_n(\alpha t).$$

(14a) $$\frac{p}{\sqrt{(p^2-\alpha^2)}}.1 = I_0(\alpha t).$$

(15) $$\frac{p}{\sqrt{(p^2+2\alpha p)}}\left(\frac{p+\alpha-\sqrt{(p^2+2\alpha p)}}{\alpha}\right)^n.1 = e^{-\alpha t}I_n(\alpha t).$$

(15a) $$\frac{p}{\sqrt{(p^2+2\alpha p)}}.1 = e^{-\alpha t}I_0(\alpha t).$$

(16) $$p\left(\frac{\sqrt{(p^2+\alpha^2)}-p}{\alpha}\right)^n.1 = \frac{n}{t}J_n(\alpha t), \qquad n>0.$$

(17) $$p\left(\frac{p-\sqrt{(p^2-\alpha^2)}}{\alpha}\right)^n.1 = \frac{n}{t}I_n(\alpha t), \qquad n>0.$$

Operations of form $\frac{1}{p+\alpha}.f(t)$

(18) $$\frac{1}{p+\alpha}.f(t) = e^{-\alpha t}\int_0^t e^{\alpha t}f(t)\,dt.$$

(19) $$\frac{1}{(p+\alpha)^n}.f(t) = e^{-\alpha t}\int_0^t\int_0^t \dots (n \text{ times})\ e^{\alpha t}f(t)\,dt\,dt\,\dots.$$

(20) $$\frac{1}{p+\alpha}.1 = \frac{1}{\alpha}(1-e^{-\alpha t}).$$

(21) $$\frac{1}{p+\alpha}.t = \frac{1}{\alpha^2}[\alpha t-(1-e^{-\alpha t})].$$

(22) $$\frac{1}{p+\alpha}.e^{-\beta t} = \frac{e^{-\beta t}-e^{-\alpha t}}{\alpha-\beta}.$$

(23) $$\frac{1}{p+\alpha}.\cos\omega t = \frac{\alpha\cos\omega t+\omega\sin\omega t-\alpha e^{-\alpha t}}{\alpha^2+\omega^2}.$$

(24) $$\frac{1}{p+\alpha}.\sin\omega t = \frac{\alpha\sin\omega t-\omega\cos\omega t+\omega e^{-\alpha t}}{\alpha^2+\omega^2}.$$

Forms (18) to (24) may be regarded as special cases of Heaviside's Shifting Theorem, namely

$$(25) \quad F(p+\alpha) \,.\, f(t) = e^{-\alpha t} F(p) \,.\, \{e^{\alpha t} f(t)\}.$$

Note. In forms (13) to (17), J_0, J_n, I_0 and I_n are the standard notations for Bessel functions, which may be found tabulated, for example, in Dale's *Five-Figure Tables of Mathematical Functions*.

Observe that (1) and (2) enable us to deal with forms which are standard except for a p more or less in the top or bottom of the p-expression. Thus, (1) in conjunction with (16) gives

$$\left(\frac{\sqrt{(p^2+\alpha^2)}-p}{\alpha}\right)^n . 1 = \int_0^t \frac{n}{t} J_n(\alpha t)\, dt.$$

APPENDIX II

THE BINOMIAL THEOREM

The expansion of $(1+x)^n$ in ascending powers of x is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!} x^2 + \frac{n(n-1)(n-2)}{3\,!} x^3 + \dots$$
$$+ \frac{n(n-1)\dots(n-m+1)}{m\,!} x^m + \dots.$$

Note that n may be negative or fractional—in fact, in operational work, it almost certainly will be. The number of terms in the series will then be infinite.

When x is a number, the statement of the series must be hedged with provisos about the permissible values of x. In our work, however, x will be an operator, such as α/p; and the validity of the method must then be judged by results. However, if the p-expression tends to a finite value when p is treated as a number and is made to tend towards infinity, binomial expansion of the p-expression in powers of $1/p$ will lead to a t-series which means something.

APPENDIX III

TRIGONOMETRICAL AND OTHER FORMULAE

This Appendix and the succeeding one are intended to provide a list of those formulae which are most likely to be useful in operational circuit work, not a complete list for all occasions. The hyperbolic functions (cosh, sinh) are therefore very scantily represented, because they can always be replaced by exponentials ($e^{\pm \alpha x}$), and the latter have been preferred in the text of the book.

(1) $$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots.$$

(2) $$\cos x = \frac{e^{jx}+e^{-jx}}{2} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots.$$

(3) $$\sin x = \frac{e^{jx}-e^{-jx}}{2j} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots.$$

(4) $$\cosh x = \frac{e^{x}+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots.$$

(5) $$\sinh x = \frac{e^{x}-e^{-x}}{2} = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\ldots.$$

(6) $$\cos^2 x+\sin^2 x = 1.$$

(7) $$\cos 2x = \cos^2 x-\sin^2 x = 2\cos^2 x-1 = 1-2\sin^2 x.$$

(8) $$\sin 2x = 2\sin x\cos x.$$

(9) $$\cos(x+y) = \cos x\cos y-\sin x\sin y.$$
$$\cos(x-y) = \cos x\cos y+\sin x\sin y.$$

(10) $$\sin(x+y) = \sin x\cos y+\cos x\sin y.$$
$$\sin(x-y) = \sin x\cos y-\cos x\sin y.$$

(11) $$a\cos x+b\sin x = \sqrt{(a^2+b^2)}\cos(x-\phi),$$
$$\text{where } \cos\phi = a/\sqrt{(a^2+b^2)}, \quad \sin\phi = b/\sqrt{(a^2+b^2)}.$$
$$a\cos x+b\sin x = \sqrt{(a^2+b^2)}\sin(x+\psi),$$
$$\text{where } \cos\psi = b/\sqrt{(a^2+b^2)}, \quad \sin\psi = a/\sqrt{(a^2+b^2)}.$$

(12) $$\cos x \cos y = \tfrac{1}{2}[\cos(x-y)+\cos(x+y)].$$

(13) $$\sin x \sin y = \tfrac{1}{2}[\cos(x-y)-\cos(x+y)].$$

(14) $$\sin x \cos y = \tfrac{1}{2}[\sin(x-y)+\sin(x+y)].$$

(15) $$\cos x+\cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}.$$

(16) $$\cos x-\cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.$$

(17) $$\sin x+\sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}.$$

(18) $$\sin x-\sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}.$$

The following Bessel function formulae may perhaps be inserted here:

(19) $$J_n(x) = \frac{(x/2)^n}{n!}\left(1-\frac{(x/2)^2}{1!(n+1)}+\frac{(x/2)^4}{2!(n+1)(n+2)}-\frac{(x/2)^6}{3!(n+1)(n+2)(n+3)}+\ldots\right).$$

(20) $$I_n(x) = \frac{(x/2)^n}{n!}\left(1+\frac{(x/2)^2}{1!(n+1)}+\frac{(x/2)^4}{2!(n+1)(n+2)}+\frac{(x/2)^6}{3!(n+1)(n+2)(n+3)}+\ldots\right).$$

(21) $$\frac{2n}{x}J_n(x) = J_{n-1}(x)+J_{n+1}(x).$$

(22) $$\frac{2n}{x}I_n(x) = I_{n-1}(x)-I_{n+1}(x).$$

APPENDIX IV

DERIVATIVES AND INTEGRALS

These lists are severely limited to functions which are useful in operational circuit work. The independent variable has been called t.

Table of Derivatives

	Function	Derivative
(1)	uv (product of two functions of t).	$u\dfrac{dv}{dt}+v\dfrac{du}{dt}$.
(2)	$f(u)$, where $u=\phi(t)$ (function of a function, like $\sin(\omega t+\theta)$).	$\dfrac{df}{du}\cdot\dfrac{du}{dt}$, or $f'(u).\phi'(t)$.
(3)	$(\alpha t)^n$.	$n\alpha^n t^{n-1}$.
(4)	$e^{\alpha t}$.	$\alpha e^{\alpha t}$.
(5)	$\cos\omega t$.	$-\omega\sin\omega t$.
(6)	$\sin\omega t$.	$\omega\cos\omega t$.
(7)	$\cosh\alpha t$.	$\alpha\sinh\alpha t$.
(8)	$\sinh\alpha t$.	$\alpha\cosh\alpha t$.
(9)	$e^{\alpha t}\cos(\omega t+\theta)$.	$\sqrt{(\alpha^2+\omega^2)}\,e^{\alpha t}\cos(\omega t+\theta+\phi)$.
	$e^{\alpha t}\sin(\omega t+\theta)$.	$\sqrt{(\alpha^2+\omega^2)}\,e^{\alpha t}\sin(\omega t+\theta+\phi)$, where $\cos\phi=\alpha/\sqrt{(\alpha^2+\omega^2)}$, $\sin\phi=\omega/\sqrt{(\alpha^2+\omega^2)}$.
(10)	$J_n(\alpha t)$.	$\begin{cases}\alpha J_{n-1}(\alpha t)-\dfrac{n}{t}J_n(\alpha t).\\ \dfrac{n}{t}J_n(\alpha t)-\alpha J_{n+1}(\alpha t).\\ \dfrac{\alpha}{2}[J_{n-1}(\alpha t)-J_{n+1}(\alpha t)].\end{cases}$
(11)	$J_0(\alpha t)$.	$-\alpha J_1(\alpha t)$.

(12) $I_n(\alpha t)$. $\begin{cases} \alpha I_{n-1}(\alpha t) - \dfrac{n}{t} I_n(\alpha t). \\ \dfrac{n}{t} I_n(\alpha t) + \alpha I_{n+1}(\alpha t). \\ \dfrac{\alpha}{2} [I_{n-1}(\alpha t) + I_{n+1}(\alpha t)]. \end{cases}$

(13) $I_0(\alpha t)$. $\alpha I_1(\alpha t)$

TABLE OF INTEGRALS

	Function	*Integral (Indefinite)*
(1)	$u\dfrac{dv}{dt}$ (product of two functions of t, one of which can be integrated).	$uv - \int v \dfrac{du}{dt}\, dt$. (Integration by Parts.)
(2)	$(\alpha t)^n$.	$\dfrac{\alpha^n t^{n+1}}{n+1}$.
(3)	$e^{\alpha t}$.	$\dfrac{e^{\alpha t}}{\alpha}$.
(4)	$\cos \omega t$.	$\dfrac{1}{\omega} \sin \omega t$.
(5)	$\sin \omega t$.	$-\dfrac{1}{\omega} \cos \omega t$.
(6)	$\cosh \alpha t$.	$\dfrac{1}{\alpha} \sinh \alpha t$.
(7)	$\sinh \alpha t$.	$\dfrac{1}{\alpha} \cosh \alpha t$.
(8)	$e^{\alpha t} \cos(\omega t + \theta)$.	$\begin{cases} \dfrac{e^{\alpha t}}{\alpha^2 + \omega^2} [\alpha \cos(\omega t + \theta) + \omega \sin(\omega t + \theta)], \\ \dfrac{e^{\alpha t}}{\sqrt{(\alpha^2 + \omega^2)}} \cos(\omega t + \theta - \phi). \end{cases}$
	$e^{\alpha t} \sin(\omega t + \theta)$.	$\begin{cases} \dfrac{e^{\alpha t}}{\alpha^2 + \omega^2} [\alpha \sin(\omega t + \theta) - \omega \cos(\omega t + \theta)], \\ \dfrac{e^{\alpha t}}{\sqrt{(\alpha^2 + \omega^2)}} \sin(\omega t + \theta - \phi), \end{cases}$

where $\cos\phi = \alpha/\sqrt{(\alpha^2 + \omega^2)}$,
$\sin\phi = \omega/\sqrt{(\alpha^2 + \omega^2)}$.

	Function	*Integral (Indefinite)*
(9)	$\cos\omega_1 t\cos\omega_2 t$. $\sin\omega_1 t\sin\omega_2 t$. $\sin\omega_1 t\cos\omega_2 t$.	Use trigonometrical formulae numbered (12), (13), (14).
(10)	$\dfrac{J_n(\alpha t)}{t^{n-1}}$.	$-\dfrac{J_{n-1}(\alpha t)}{\alpha t^{n-1}}$.
(11)	$\dfrac{I_n(\alpha t)}{t^{n-1}}$.	$\dfrac{I_{n-1}(\alpha t)}{\alpha t^{n-1}}$.
(12)	$J_n(\alpha t)$.	$\dfrac{2}{\alpha}\{J_{n+1}(\alpha t)+J_{n+3}(\alpha t)$ $+J_{n+5}(\alpha t)+\dots\text{ ad inf.}\}$.
(13)	$I_n(\alpha t)$.	$\dfrac{2}{\alpha}\{I_{n+1}(\alpha t)-I_{n+3}(\alpha t)$ $+I_{n+5}(\alpha t)-\dots\text{ ad inf.}\}$.

APPENDIX V

THE SOLUTION AND PROPERTIES OF ALGEBRAIC EQUATIONS

This is a bare summary of the properties of algebraic equations—just enough to enable the reader to find the factors of his p-expressions. For proofs of the assertions which are here dogmatically made, he should consult some such book as F. W. Turnbull's *Theory of Equations*.

Definitions

An expression of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + lx + m,$$

which (for brevity) we shall call $F(x)$, is said to be a *polynomial of degree n* in x. $a, b, \dots, m$ are constants, known as the *coefficients* of the various powers of x. We shall assume that these coefficients are real, and that a is positive.

Any value of x which makes $F(x) = 0$ is called a *root* of the equation $F(x) = 0$.

Properties of Equations

(1) An equation of degree n has n roots, which may be real or complex. Two or more roots may be the same.

(2) When the roots $x_1, x_2, \dots, x_n$ are known, $F(x)$ may be written as the product of n factors, thus:

$$ax^n + bx^{n-1} + \dots + lx + m \equiv a(x - x_1)(x - x_2)\dots(x - x_n).$$

(3) This equation demonstrates the relations which exist between the coefficients $a, b, \dots, m$ and the roots $x_1, x_2, \dots, x_n$; namely

$$\begin{aligned} b/a &= -(x_1 + x_2 + \dots + x_n) \\ &= -(\text{Sum of roots}). \end{aligned}$$

$$\begin{aligned} c/a &= +(x_1x_2 + x_1x_3 + \dots + x_2x_3 + \dots) \\ &= +(\text{Sum of all possible products of pairs of roots}). \end{aligned}$$

....

$$\begin{aligned} m/a &= (-1)^n x_1x_2\dots x_n \\ &= (-1)^n \times (\text{Product of roots}). \end{aligned}$$

(4) It follows that if we change alternate signs in $F(x)$, thus:

$$ax^n - bx^{n-1} + cx^{n-2} - dx^{n-3} + \ldots + (-1)^n m = 0,$$

we obtain an equation whose roots are numerically the same as the roots of $F(x) = 0$, but with their signs changed.

(5) It also follows that if we multiply the coefficients by $1, \lambda, \lambda^2, \ldots$, thus:

$$ax^n + b\lambda x^{n-1} + c\lambda^2 x^{n-2} + \ldots + m\lambda^n = 0,$$

we obtain an equation whose roots are λx_1, λx_2, λx_3, ..., etc.

(6) Since the coefficients are assumed to be real, the complex roots (if any) can only occur in conjugate pairs, $\beta \pm j\gamma$.

(7) Hence any equation of odd degree must have at least one real root.

(8) Hence also $F(x)$ can always be split into real factors, of degree not greater than 2.

(9) If $F(x)$ is positive when x is replaced by one number q, but negative when x is put equal to another number r, then the equation $F(x) = 0$ has at least one real root between q and r.

The last theorem affords a means of quickly finding the approximate values of the real roots of an equation. For example, consider

$$x^3 - 2x - 5 = 0,$$

an equation which has become traditional as an instance in the theory of equations, ever since it was first propounded by John Wallis (1616–1703), of Oxford. Let us jot down the values of $F(x)$ for a few simple values of x:

x	$-\infty$	-3	-2	-1	0	$+1$	$+2$	$+3$	$+\infty$
$F(x)$	$-\infty$	-26	-9	-4	-5	-6	-1	$+16$	$+\infty$

It will be seen that there is at least one root between 2 and 3, seemingly much nearer to 2. Actually this equation has only one real root, viz. 2·09455....

Horner's Method of obtaining Real Roots

Having got somewhere near the root in this way, we can get as many decimal places as we require by the following method; some people are afraid of it, but its terrors are much exaggerated. It will be explained with reference to a cubic equation, but it is applicable to equations of any degree.

Suppose that inspection has shown the equation

$$ax^3+bx^2+cx+d=0$$

to possess a root between 1 and 10, whose first digit is n. The coefficients are first written with their appropriate signs in a horizontal row, followed by the first digit of the root:

$$\begin{array}{ccccc} a & b & c & d & (\,n \end{array}$$

The first coefficient a is now multiplied by n, and the product added to b; this sum is multiplied by n, and the product added to c; and so on, thus:

$$\begin{array}{ccccc} a & b & c & d & (\,n \\ & na & nA & nB & \\ \cline{2-4} & \text{(Sum) } A & \text{(Sum) } B & \text{(Sum) } C & \end{array}$$

The process is repeated, stopping short at the last stage but one; then again, stopping short at the last stage but two; and so on, until the last process is but a single addition of na. The calculation for the cubic equation now looks like this:

$$\begin{array}{ccccc} a & b & c & d & (\,n \\ & na & nA & nB & \\ \cline{2-4} & A & B & C & \\ & na & nD & & \\ \cline{2-3} & D & E & & \\ & na & & & \\ \cline{1-2} & F & & & \end{array}$$

The lowest digit in each column is usually brought into prominence by a dividing line ruled as shown; these constitute a new set of coefficients, namely a, F, E, C.

It can be shown that the equation

$$ax^3+Fx^2+Ex+C=0$$

has roots which are smaller by n than the corresponding roots of the original equation; so that a root of the original equation having the first digit n has a counterpart root in the new equation which is less than 1.

The equation

$$ax^3 + 10Fx^2 + 100Ex + 1000C = 0$$

has all its roots 10 times as large as the corresponding roots of

$$ax^3 + Fx^2 + Ex + C = 0 \text{ (Theorem (5) of the last paragraph).}$$

If we multiply the coefficients of the derived equation by 1, 10, 100, ..., in this way, we obtain an equation in which the root we are chasing (or its counterpart) again lies between 1 and 10. The whole process can be repeated, to obtain the second digit of the root of the original equation.

The following numerical examples show how the work should be set out.

Examples of Horner's Method

Example 1. To find that root of $2x^3 - 5x^2 + 1 = 0$, which lies between 2 and 3.

2	−5	0	1	(2·41
	4	−2	−4	
	−1	−2	−3000	
	4	6	2848	
	3	400	− 152000	
	4	312	106542	
	70	712	− 45458	
	8	344		
	78	105600		
	8	942		
	86	106542		
	8	944		
	940	107486		
	2			
	942			
	2			
	944			
	2			
	946			

From this stage, several more figures may be quickly obtained by a process analogous to the 'contracted division' of elementary

arithmetic. Instead of adding 0's to the coefficients, we strike off one, two, three, ... digits, but take account of the last digit neglected. The example therefore continues as follows:

//~~2~~	9~~46~~	10748~~6~~	−45458	(2·414214
		37~~8~~	43146	
		10786~~4~~	− 2312	
		37~~8~~	2165	
	/~~9~~	1082~~4~~	− 147	
		~~2~~	108	
		1082~~6~~	− 39	
		~~2~~	43	
		10~~83~~		

This result is correct to six decimal places.

The degree of accuracy here obtained is greater than is usually needful in engineering problems. It has been thought well to go to this accuracy, in order to present a thorough illustration; but much less arithmetic is involved if the contracted method is begun after two stages, and the result is only wrong by 1 part in 24,000:

2	−5	0	1	(2·4143
	4	−2	−4	
	−1	−2	−3000	
	4	6	2848	
	3	400	− 152	
	4	312	106	
	70	712	− 46	
	8	344	43	
	78	105~~6~~	− 3	
	8	~~9~~	3	
	86	106~~5~~		
	8	~~9~~		
//~~2~~	//~~94~~	1~~07~~		

Example 2. In § 7·4, we found it necessary to solve the equation

$$216x^3 - 162x^2 - 60x + 47 = 0.$$

From the way in which the equation was derived, it appeared that we were interested only in a real root greater than $\frac{25}{36}$, or 0·694.

The usual method of root-location shows that the equation has one negative and two positive roots; both positive roots are less than 1, but one is a little greater than 0·7, while the other is less than 0·6. We therefore write down the equation whose roots are those of the original equation multiplied by 10, namely

$$216x^3 - 1620x^2 - 6000x + 47000 = 0,$$

and seek that root whose first digit is 7.

The Horner calculation, using the contracted method after two stages, looks like this (the coefficients have all been divided by 4):

/ $\cancel{5}\cancel{4}$	−405	−1500	11750	(0·7087
	378	− 189	−11823	
	− 27	−1689	−73000	
	378	2457	66131	
	351	$7680\cancel{0}$	− 6869	
	378	$586\cancel{4}$	6236	
	$72\cancel{9}\cancel{0}$	$8266\cancel{4}$	− 633	
	$\cancel{4}$	$589\cancel{6}$		
	$73\cancel{3}$	$885\cancel{6}$		
	$\cancel{4}$	$5\cancel{2}$		
	$73\cancel{7}$	$890\cancel{8}$		
	$\cancel{4}$			
	$\cancel{7}\cancel{4}$			

This degree of accuracy is sufficient. If the reader follows the working through for himself, he will easily discover what happens at the stage when the digit is zero.

The Mechanism of Horner's Method

The process of solution will be much better understood if Example 1 is further considered.

The roots of the equation are actually

$$-0{\cdot}5858, \quad +0{\cdot}5000, \quad +2{\cdot}4142,$$

to four places of decimals.

The first reduction makes them

$$-2{\cdot}5858, \quad -1{\cdot}5000, \quad +0{\cdot}4142.$$

(Notice that the product has changed sign. This is to be seen in the last coefficient of the equations.)

The first multiplication gives

$$-25{\cdot}858, \quad -15{\cdot}000, \quad +4{\cdot}142.$$

The second reduction gives

$$-29{\cdot}858, \quad -19{\cdot}000, \quad +0{\cdot}142.$$

The second multiplication gives

$$-298{\cdot}58, \quad -190{\cdot}00, \quad +1{\cdot}42, \text{ and so on.}$$

It is apparent that the root which we are chasing is rapidly becoming by far the smallest root of the equation, and the reduction process cannot cause any of the roots to change their signs. This gives us a rule for determining the successive digits (after the first, which was found by location of the roots): Find the largest digit which does not cause the last coefficient to change sign. (In exceptional cases, where there are two roots very close together, this may break down; the reader can try out hypothetical cases to test this. The root-locating process may then have to be repeated.)

To find a negative root, change the signs of alternate coefficients, so that the root becomes positive.

To find a root not lying between 1 and 10, multiply or divide all roots by such a multiple of 10 as will bring it within those limits.

The Quadratic Equation

The roots of the quadratic equation

$$ax^2+bx+c = 0$$

may be found immediately from the formula

$$x_1, x_2 = \frac{-b \pm \sqrt{(b^2-4ac)}}{2a},$$

which is given in elementary algebra text-books.

The Cubic Equation

Every cubic equation must have at least one real root. This may be found by Horner's method, and the equation reduced to a quadratic by dividing by the corresponding factor.

THE QUARTIC (BIQUADRATIC) EQUATION

A quartic equation may have four real roots, or two, or none. The last case presents us with a new difficulty, which may be removed in the following manner.

We know that it must be possible, in at least one way, to resolve an expression of form

$$ax^4+bx^3+cx^2+dx+e$$

into real quadratic factors. Therefore it must be possible to find real numbers f, g, h, such that

$$ax^4+bx^3+cx^2+dx+e \equiv a\left\{\left(x^2+\frac{b}{2a}x+f\right)^2-(gx+h)^2\right\}.$$

(Note that the coefficient of x in the first squared expression has been assigned that value which leads, on squaring, to the right coefficient for x^3.)

Equating coefficients, we find

$$\frac{c}{a}=\left(\frac{b}{2a}\right)^2+2f-g^2,$$

$$\frac{d}{a}=\frac{fb}{a}-2gh,$$

$$\frac{e}{a}=f^2-h^2.$$

The first equation gives $g^2=\dfrac{b^2}{4a^2}-\dfrac{c}{a}+2f;$

the third, $h^2=f^2-\dfrac{e}{a}.$

Substituting in the second,

$$\begin{aligned}(bf-d)^2 &= 4a^2g^2h^2\\ &= 4a^2\left\{2f+\left(\frac{b^2-4ac}{4a^2}\right)\right\}\left\{f^2-\frac{e}{a}\right\},\end{aligned}$$

which is a cubic equation in f, and therefore within our compass. An example of this method will be found worked out in § 7·4.

The Quintic Equation

Every quintic equation must have at least one real root, and can therefore be reduced to a quartic.

Equations of Higher Order

The sextic equation may have three pairs of complex roots. If it has, the methods so far given all break down. Methods are known for overcoming this obstacle, but an engineer is unlikely to find himself repaid for the labour of learning or of applying them; therefore I deliberately refrain from carrying this discussion further. Interested readers will find the usual method described in *Heaviside's Operational Calculus*, by E. J. Berg; it is called Graeffe's method.

Conditions for Real or Complex Roots

(*a*) *Quadratic*, $ax^2+bx+c=0.$

If $$b^2-4ac>0,$$

the roots are real; if this expression is less than 0, they are complex.

(*b*) *Cubic*, $ax^3+bx^2+cx+d=0.$

If $$27a^2d^2-b^2c^2+4ac^3+4db^3-18abcd<0,$$

all the roots are real; if this expression is greater than 0, there is one real root and a pair of complex roots.

(*c*) *Biquadratic*, $ax^4+bx^3+cx^2+dx+e=0.$

If $$4(12ae-3bd+c^2)^3 - (72ace+9bcd-27ad^2-27eb^2-2c^3)^2<0,$$

the equation has two roots real and two complex; if this expression is greater than 0, the roots are all complex unless both the following conditions are also satisfied:

$$8ac-3b^2<0,$$

$$16a^2(12ae-3bd+c^2)-(8ac-3b^2)^2<0;$$

in which case the roots are all real.

Conditions for Stability

The following conditions must all be satisfied if the equations are to have no roots with a positive real part. Applied to the bottom line of a p-operator, these conditions are sufficient to ensure that terms like $e^{\alpha t}$ or $e^{\beta t}\cos\omega t$ shall not occur. They are given by E. J. Routh, *Advanced Rigid Dynamics*, pp. 192–202 (1892 edition), and are usually known as 'Routh's Criteria of Stability'.

It is assumed that the first coefficient a is positive.

(*a*) *Quadratic*, $ax^2+bx+c=0.$

Conditions for stability are

$$b>0, \quad c>0.$$

(*b*) *Cubic*, $ax^3+bx^2+cx+d=0.$

Conditions for stability are

$$b>0, \quad bc-ad>0, \quad d>0.$$

(*c*) *Biquadratic*, $ax^4+bx^3+cx^2+dx+e=0.$

Conditions for stability are

$$b>0, \quad bc-ad>0, \quad bcd-ad^2-eb^2>0, \quad e>0.$$

Conditions for Existence of Complex Roots of Form $-\omega(1\pm j)$

These are the conditions discussed at the end of § 6·8; applied to the bottom line of a p-operator, they foreshadow the presence of oscillatory terms like $e^{-\omega t}\cos\omega t$ or $e^{-\omega t}\sin\omega t$.

(*a*) *Quadratic*, $ax^2+bx+c=0.$

The condition is

$$b^2-2ac=0.$$

(*b*) *Cubic*, $ax^3+bx^2+cx+d=0.$

The condition is

$$a^2d^2-b^2c^2+2ac^3+2db^3-4abcd=0.$$

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(a) CLASSICS

In the following books and papers, the fundamental theory of the subject was first set forth.

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— *Electromagnetic Theory*, 3 vols. (Reprinted by Benn.)

— *Electrical Papers*, 2 vols.

T. J. I'A. BROMWICH. 'Normal Coordinates in Dynamical Systems.' (*Proc. Lond. Math. Soc.* (2), **15**, 1916, p. 401.)

K. W. WAGNER. 'Über eine Formel Heaviside zur Berechnung von Einschaltvorgängen.' (*Archiv für Elektrotechnik*, **4**, 1916, p. 159.)

The contributions of each of these authors to the subject are mentioned in Chapter I.

(b) MODERN EXPOSITIONS

Most of these books and papers carry the subject further than the present work, in that they apply the method to continuous systems as well as to lumped systems. They have been divided into three categories, according to the mathematical knowledge which they demand of the reader. Category (1) contains works in which the mathematical aspect is subordinated to the practical. For category (2) a good knowledge of differential and integral calculus is needed, but only in application to functions of a real variable. Category (3) assumes knowledge of functions of a complex variable. The works have also been arranged within each category in approximate order of difficulty. The list does not pretend to be exhaustive.

CATEGORY (1)

E. J. BERG. *Heaviside's Operational Calculus.* (McGraw Hill, New York.)

V. BUSH. *Operational Circuit Analysis.* (John Wiley and Son, New York.)

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